Professor Mac Cullagh gave an account of his researches in the Theory of Surfaces of the second Order, in connexion with a former communication which he had made to the Academy on the same subject. These researches are contained in the following paper.

On the Surfaces of the Second Order.

There is hardly any geometrical theory which more requires to be studied, or which promises to reward better whatever thought may be bestowed upon it, than that of the surfaces of the second order. My attention was drawn to it, many years ago, by the consideration of mechanical and physical questions. In the dynamical problem of the Rotation of a Solid Body, and in the investigation of the properties of the Wave-Surface of Fresnel, I found, so long since as the year 1829, that the ellipsoid could be employed with very great advantage; while the discussion of these questions, but especially of the former,* suggested properties of the ellipsoid and its kindred surfaces which I might not otherwise have perceived. In this manner I was led to consider systems of confocal surfaces, and thence to notice the focal curves, which I discovered to be analogous, in the theory of the surfaces of the second order, to the foci in that of the plane conic sections. That theory now began to interest me on its own account, and, guided by analogy, I struck out the leading properties possessed by the surfaces in relation to their focal curves; but the interference of other matters prevented me from continuing the inquiry. I had done enough, however, in this and other parts of the theory, to open new views respecting

* The Theory of Rotation, here spoken of, was completed in the year 1831; but, from causes which need not be mentioned at present, it was not published. The investigations relative to Fresnel's Wave-Surface will be found in the Transactions of the Royal Irish Academy, vol. xvi. p. 65; vol. xvii. p. 241. See also vol. xxi. p. 32, of the same Transactions.

it; and the results at which I had arrived seemed so fitted for instruction, that when I was appointed Professor of Mathematics in the University, I made them the subject of the first lectures which I gave in that capacity, in the beginning of the year 1836. Next year the heads of these lectures were communicated to this Academy, in a paper of which a very short abstract appeared in the Proceedings.* The subject soon became a favourite one among the more advanced students in the University, who are, for the most part, excellent geometers, and in the present Article very little will be found which is not well known amongst them; very little, indeed, which was not communicated to the Academy on the occasion just mentioned, or which may not be gathered, in the shape of detached questions, out of the Examination-Papers published yearly in the University Calendar. But as nothing has yet been published on the subject in a connected form, except the brief notice in the Proceedings of the Academy, and as mathematicians in other countries attach some importance to researches of this kind, and appear to be in quest of certain principles which are familiar to us here, it seems proper to collect together the chief results that have already been obtained, in order that persons wishing to pursue these speculations may be better able to judge where their inquiries should begin, and in what direction further progress is most likely to be made.

PART I.—GENERATION OF SURFACES OF THE SECOND ORDER.

§ 1. The different species of surfaces of the second order are obtained, as is usually shown in elementary treatises, by the discussion of the general equation of the second degree among three coordinates; but it is necessary that we should also be able to derive these surfaces from a common geometrical origin, if we would bring them completely within

* Proceedings of the Royal Irish Academy, vol. i. p. 89.

the grasp of geometry. Now as the different conic sections may (with the exception of the circle) be described *in plano* by the motion of a point whose distance from a given point bears a constant ratio to its distance from a given right line,* it is natural to suppose that there must be some analogous method by which the surfaces of the second order may be generated in space. Accordingly I have sought for such a method, and I have found that (with certain analogous exceptions) every surface of the second order may be regarded as the locus of a point whose distance from a given point bears a constant ratio to its distance from a given right line, provided the latter distance be measured parallel to a given plane; this plane being, in general, oblique to the right line. The given point I call, from analogy, a *focus*, and the given right line a *directrix;* the given plane may be called a *directive plane*, and the constant ratio may be termed the *modulus*.

To find the equation of the surface so defined, let the axis of z be parallel to the directrix; let the plane of xy pass through the focus, and cut the directrix perpendicularly in Δ, the coordinates being rectangular, and their origin arbitrarily assumed in that plane; and let the axis of y be parallel to the intersection of the plane of xy with the directive plane, the angle between the two planes being denoted by ϕ. Then if we put x_1, y_1 for the coordinates of the focus, and x_2, y_2 for those of the point Δ, while the coordinates of a point S upon the surface are denoted by x, y, z, the distance of this last point from the focus will be the square root of the quantity

$$(x-x_1)^2 + (y-y_1)^2 + z^2;$$

and if a plane drawn through S, parallel to the directive plane, be conceived to cut the directrix in D, the distance SD will be the square root of the quantity

* This method of describing the conic sections is due to the Greek geometers. It is given by Pappus at the end of the Seventh Book of his *Mathematical Collections*.

$$(x-x_2)^2 \sec^2\phi + (y-y_2)^2;$$

so that, m being the modulus, the locus of the point S will be a surface of the second order, represented by the equation

$$(x-x_1)^2 + (y-y_1)^2 + z^2 = m^2\{(x-x_2)^2\sec^2\phi + (y-y_2)^2\}, \quad (1)$$

which, by making

$$\begin{aligned} \text{A} &= 1-m^2\sec^2\phi, \qquad \text{B} = 1-m^2, \\ \text{G} &= m^2x_2\sec^2\phi - x_1, \quad \text{H} = m^2y_2 - y_1, \\ \text{K} &= m^2(x_2^2\sec^2\phi + y_2^2) - x_1^2 - y_1^2, \end{aligned} \quad (2)$$

may be put under the form

$$\text{A}x^2 + \text{B}y^2 + z^2 + 2\text{G}x + 2\text{H}y = \text{K}, \quad (3)$$

showing that the plane of xy is one of the principal planes of the surface, and that the planes of xz and yz are parallel to principal planes.

Before we proceed to discuss this equation, it may be well to observe that as it remains the same when ϕ is changed into $-\phi$, or into $180° - \phi$, the directive plane may have two positions equally inclined to the plane of xy, and therefore equally inclined to the directrix. Indeed it is obvious that, if through the point S we draw two planes making equal angles with the directrix, and cutting it in the points D and D′ respectively, the distances SD and SD′ will be equal. Every surface described in this way has consequently two directive planes; and as each of these planes is parallel to the axis of y, their intersection is always parallel to one of the axes of the surface. This axis may therefore be called the *directive axis*. The directive planes have a remarkable relation to the surface, as may be shown in the following manner:—

Suppose a section of the surface to be made by a plane which is parallel to one of the directive planes, and which cuts the directrix in D; then the distance of any point S of the section from the focus F will have a constant ratio to its distance SD from the point D; and, as the locus of a point

whose distances from the two points F and D are in a constant ratio to each other, is a plane or a sphere, according as the ratio is one of equality or not, it follows that the section aforesaid will be a right line in the one case, and a circle in the other. Hence it appears that all *directive sections*, that is, all sections made in the surface by planes parallel to either of the directive planes, are *right lines* when the modulus is unity, and *circles* when the modulus is different from unity.

Since the equation (3) is not altered by changing the sign of ϕ, or by changing ϕ into its supplement, we may suppose this angle (when it is not zero) to be always positive and less than 90°; for the supposition $\phi = 90°$ is to be excluded, as it would make the secant of ϕ infinite, and the directive planes parallel to the directrix. In the discussion of the equation there are two leading cases to be considered, answering to two classes of surfaces. The first case, when neither A nor B vanishes, gives the ellipsoid, the two hyperboloids, and the cone; the second, when either or each of these quantities is zero, includes the two paraboloids and the different kinds of cylinders.

§ 2. *First Class of Surfaces.*—When neither A nor B vanishes, we may make both G and H vanish, by properly assuming the origin of coordinates. Supposing this done, we have

$$x_1 = m^2 x_2 \sec^2\phi, \qquad y_1 = m^2 y_2, \tag{4}$$

the equation of the surface being then

$$\mathrm{A}x^2 + \mathrm{B}y^2 + z^2 = \mathrm{K}, \tag{5}$$

in which the axes of coordinates are of course the axes of the surface. When K is not zero, the surface is an ellipsoid or hyperboloid, having its centre at the origin of coordinates; when $\mathrm{K} = 0$, the surface is a cone having its vertex at the origin.

Eliminating x_2, y_2 from the value of K, by means of the relations (4), we get

$$\mathrm{K} = \frac{\mathrm{A}}{1-\mathrm{A}} x_1^2 + \frac{\mathrm{B}}{1-\mathrm{B}} y_1^2; \tag{6}$$

and eliminating x_1, y_1 in like manner, we get

$$\text{K} = \text{A}(1-\text{A})x_2^2 + \text{B}(1-\text{B})y_2^2; \tag{7}$$

from which expressions it appears that, every thing else remaining, the focus and directrix may be changed without changing the surface described. For in order that the surface may remain unchanged, it is only necessary that K should remain constant, since A and B are supposed constant. This condition being fulfilled, the focus may be any point F whose coordinates x_1, y_1 satisfy the equation (6), and Δ (the foot of the directrix) may be any point whose coordinates x_2, y_2 satisfy the equation (7); it being understood, however, that when one of these points is chosen, the other is determined. The locus of F (supposing K not to vanish) is therefore an ellipse or a hyperbola,* which may be called the *focal curve*, or the *focal line;* and the locus of Δ is another ellipse or hyperbola, which may be called the *dirigent* curve or line: the centre of each curve is the centre of the surface, and its axes coincide with the axes of the surface which lie in the plane of xy. Moreover, as the quantities $1 - \text{A}$ and $1 - \text{B}$ are essentially positive, the two curves are always of the same kind, that is, both ellipses, or both hyperbolas; and when they are hyperbolas, their real axes have the same direction. The directrix, remaining always parallel to the axis of z, describes a cylinder which may be called the *dirigent cylinder*.

Since, by the relations (4), the corresponding coordinates of F and Δ have always the same sign, these points either lie within the same right angle made by the axes of x and y, or lie on the same axis, at the same side of the centre. And as these relations give

$$x_2 - x_1 = \frac{\text{A}}{1-\text{A}}x_1, \qquad y_2 - y_1 = \frac{\text{B}}{1-\text{B}}y_1, \tag{8}$$

* In the Proceedings of the Academy, vol. i. p. 90, it was stated inadvertently that "if we confine ourselves to the central surfaces, the locus of the foci will be an ellipse."

it is easy to see that the right line ΔF is a normal to the focal curve; for the quantities $x_2 - x_1$ and $y_2 - y_1$ are proportional to the cosines of the angles which that right line makes with the axes of x and y respectively, while the values just given for these quantities are, in virtue of the equation (6), proportional to the cosines of the angles which the normal to the focal curve at the point F makes with the same axes.

It may also be shown that if the directrix prolonged through Δ intersect a directive plane in a certain point, and if a right line drawn through F, parallel to the directrix, intersect the same plane in another point, the right line joining those points will be a normal to the curve described in that plane by the first point.

§ 3. To find in what way the focal and dirigent curves are connected with the surface, let the equations (5), (6), (7) (when K does not vanish) be put under the forms

$$\frac{x^2}{P} + \frac{y^2}{Q} + \frac{z^2}{R} = 1, \tag{9}$$

$$\frac{x_1^2}{P_1} + \frac{y_1^2}{Q_1} = 1, \qquad \frac{x_2^2}{P_2} + \frac{y_2^2}{Q_2} = 1, \tag{10}$$

so that the quantities P, Q, R may represent the squares of the semiaxes of the surface, and P_1, Q_1, P_2, Q_2 the squares of the semiaxes of the curves, these quantities being positive or negative, according as the corresponding semiaxes are real or imaginary. Then we have

$$\begin{aligned} &P = \frac{K}{A}, \qquad Q = \frac{K}{B}, \qquad R = K, \\ &P_1 = P(1 - A), \qquad Q_1 = Q(1 - B), \\ &P_2 = \frac{P}{1 - A}, \qquad Q_2 = \frac{Q}{1 - B}; \end{aligned} \tag{11}$$

whence it follows that

$$P_1 P_2 = P^2, \qquad Q_1 Q_2 = Q^2, \tag{12}$$

and also that

$$P_1 = P - R, \qquad Q_1 = Q - R. \tag{13}$$

From equations (12) we see that P_1 and P_2 have always the same sign, as also Q_1 and Q_2; and that, neglecting signs, the semiaxes of the surface are mean proportionals between the corresponding semiaxes of the focal and dirigent curves. These curves are therefore reciprocal polars with respect to the section made in the surface by the plane of xy; and it would be easy to show that the points F and Δ are reciprocal points, or that a tangent applied at one of them to the curve which is its locus has the other for its pole.

The focal curve, when we know in which of the principal planes it lies, is determined by the conditions (13), and as it depends on the relative magnitudes of the quantities P, Q, R, it will be convenient to distinguish the axes of the surface, with relation to these magnitudes. Supposing, therefore, the quantities P, Q, R to be taken with their proper signs, as they are in the equation (9), that axis to which the greatest of them (which is always positive) refers, shall be called the *primary* axis; and that to which the quantity algebraically least has reference, shall be termed the *secondary* axis; while the quantity which has an intermediate algebraic value shall mark the *middle* or *mean* axis. Then, since both P_1 and Q_1 will be negative, if R be the greatest of the quantities aforesaid, the focal curve cannot lie in the plane of the mean and secondary axes. Its plane must therefore pass through the primary axis; it will be the plane of the primary and mean axes, if R be the least of the three quantities; but the plane of the primary and secondary axes, if R be the intermediate quantity. In the former case the curve will be an ellipse, in the latter a hyperbola; and we shall extend the name of focal curves to both the curves so determined, though it may happen that only one of them can be used in the generation of the surface by the *modular method*, as the method of which we are treating may be

called, from its employment of the modulus. A focal curve which can be so used shall be distinguished as a *modular focal;* but each focal, whether modular or not, shall be supposed to have a dirigent curve and a dirigent cylinder connected with it by the relations already laid down.

Since $P_1 - Q_1 = P - Q$, the foci of a focal curve are the same as those of the principal section in the plane of which it lies, and they are therefore on the primary axis of the surface. It will sometimes contribute to brevity of expression, if we also give the name of primary to the major axis of an ellipse and to the real axis of a hyperbola. We may then say that the primary axes of the surface and of its two focal curves are coincident in direction; and that (as is evident) the foci of either curve are the extremities of the primary axis of the other.

If K be supposed to approach gradually to zero, while A and B remain constant, the focal and dirigent ellipses will gradually contract, and the focal and dirigent hyperbolas will approach to their asymptotes, which remain fixed. When K actually vanishes, the surface becomes a cone; the two ellipses are each reduced to a point coinciding with the vertex of the cone, and each hyperbola is reduced to the pair of right lines which were previously the asymptotes. The dirigent cylinder, in the one case, is narrowed into a right line; in the other case it is converted into a pair of planes, which we may call the *dirigent planes* of the cone.

§ 4. We have now to show how the different kinds of surfaces belonging to the first class are produced, according to the different values of the modulus and other constants concerned in their generation.

I. When m is less than $\cos\phi$, the quantities A, B, K, P, Q, R are all positive, and Q is intermediate in value between P and R. The surface is therefore an ellipsoid, and its mean axis is the directive. As the quantities $1 - A$ and $1 - B$ are always positive, the focal and dirigent curves are ellipses.

Here we cannot suppose K to vanish, as the surface would then be reduced to a point.

When $\phi = 0$, that is, when the directive planes coincide with each other, and therefore with a plane perpendicular to the directrix, so that SD is the shortest distance of the point S from the directrix, the surface is a spheroid produced by the revolution of an ellipse round its minor axis, and the focal and dirigent curves are circles.

II. When m is greater than unity, A and B are negative; and if K be finite, it is also negative; whence P and Q are positive, and R is negative. Also, supposing ϕ not to vanish, Q is greater than P. The surface is therefore a hyperboloid of one sheet, with its real axes in the plane of xy; and the directive axis is the primary. The focal and dirigent curves are ellipses. But when $\phi = 0$, the surface is that produced by the revolution of a hyperbola round its imaginary axis, and the focal and dirigent* are circles.

If $K = 0$, which implies, since A and B have the same sign, that x_1, y_1, x_2, y_2 are each zero, the surface is a cone having the axis of z for its internal axis; and the focal and dirigent are each reduced to a point. The focus and directrix are consequently unique; the focus can only be the vertex of the cone, the directrix can only be the internal axis; and the directrix therefore passes through the focus. The directive axis, which coincides with the axis of y, is one of the external axes; that one, namely, which is parallel to the greater axes of the elliptic sections made in the cone by planes perpendicular to its internal axis. This is on the supposition that ϕ is finite; for, when $\phi = 0$, the cone becomes one of revolution round the axis of z.

III. When m is greater than cos ϕ, but less than unity, we have A positive and B negative, and the species of the

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* When the term *dirigent* stands alone, it is understood to mean a dirigent *line*.

surface depends on K. It is inconsistent with these conditions to suppose $\phi = 0$, and therefore the surface cannot, in this case, be one of revolution. The value of K may be supposed to be given by the formula

$$K = \frac{1 - A}{A}(x_2 - x_1)^2 + \frac{1 - B}{B}(y_2 - y_1)^2, \qquad (14)$$

which contains only the relative coordinates of the focus and the foot of the directrix, and is a consequence of the equations (6) and (8).

1°. If K is a positive quantity, the surface is a hyperboloid of one sheet, with its secondary axis in the direction of x; the primary axis, as before, is the directive, but the focal and dirigent are now hyperbolas.

2°. If K is a negative quantity, the surface is a hyperboloid of two sheets, having its primary axis coincident with that of x. The secondary axis is the directive; the focal and dirigent are hyperbolas.

3°. If $K = 0$, the surface is a cone, having the axis of x for its internal axis; the directive axis being, as before, that external axis to which the greater axes of the elliptic sections, made by planes perpendicular to the internal axis, are parallel. The axis of z is the other external axis, which may be called the *mean axis* of the cone, because it coincides with the mean axis of any hyperboloid to which the cone is asymptotic. As A and B have different signs, it is evident, from the equations (6) and (7), that the focal and dirigent are each a pair of right lines passing through the vertex, each pair making equal angles with the internal axis. Two planes, each of which is drawn through the mean axis and a dirigent line, are the dirigent planes of the cone.

The corresponding focal and dirigent lines are those which lie within the same right angle made by the internal and directive axes; and since by the equations (6) and (8) the value of K may be written

$$\kappa = x_1 (x_2 - x_1) + y_1 (y_2 - y_1), \tag{15}$$

we see that, as κ now vanishes, the right line joining corresponding points F and Δ upon these lines is perpendicular to the focal line. Of the two sides of the cone which are in the plane of xy, one lies between each focal and its dirigent; and it may be inferred from the equations, that the tangents of the angles which the internal axis makes with a focal line, with one of these sides of the cone, and with a dirigent line, are in continued proportion, the proportion being that of the cosine of ϕ to unity. And hence it follows, that these two sides of the cone, with a focal line and its dirigent, cut harmonically any right line which crosses them.

§ 5. From this discussion it appears, that the ellipsoid and the hyperboloid of two sheets can be generated modularly, each in one way only, the modular focal being the ellipse for the former, and the hyperbola for the latter; but that the hyperboloid of one sheet can be generated in two ways, each of its focals being modular, and each focal having its proper modulus. The cone also admits two modes of generation,* in one of which, however, the focus is limited to the vertex of the cone, and the directrix to its internal axis.

* The double generation of the cone, when its vertex is the focus, may be proved synthetically by the method indicated in the Examination Papers of the year 1838, p. xlvi (published in the University Calendar for 1839). Supposing the cone to stand on a circular base (one of its directive sections), and to be circumscribed by a sphere, the right lines joining its vertex with the two points where a diameter perpendicular to the plane of the base intersects the sphere, will be its internal and mean axes. Then if P be either of these points, V the vertex, C the point where the axis PV cuts the plane of the base, and B any point in the circumference of the base, the triangles PVB and PBC will be similar, since the angles at V and B are equal, and the angle at P is common to both triangles; therefore BV will be to BC as PV to PB, that is, in a constant ratio. It is not difficult to complete the demonstration, when the focus is supposed to be any point on one of the focal lines.

But when the hyperboloid of one sheet, or the cone, is a surface of revolution, it has only one mode of modular generation. In cases of double generation, the directive planes of course remain the same, as they have a fixed relation to the surface. A modular focal, it may be observed (and the remark applies equally to surfaces of the second class), is distinguished by the circumstance that it does not intersect the surface. The only exception to this rule are the focal lines of the cone, which pass through its vertex. A focal which is not modular may be called *umbilicar*, because it intersects the surface in the umbilics; an umbilic being a point on the surface where the tangent plane is parallel to a directive plane. Thus the focal hyperbola of the ellipsoid, and the focal ellipse of the hyperboloid of two sheets, are umbilicar focals, and pass through the umbilics of these surfaces; but the hyperboloid of one sheet has no umbilics, and accordingly both its focals are modular, and neither of them intersects the surface. The umbilicar focals and dirigents have properties which shall be mentioned hereafter.

An umbilicar focal and the principal section whose plane coincides with that of the focal are curves of different kinds, the one being an ellipse when the other is a hyperbola; but a modular focal is always of the same kind with the coincident section of the surface, being an ellipse, a hyperbola, or a pair of right lines, according as the section is an ellipse, a hyperbola, or a pair of right lines; and when the section is reduced to a point, so likewise is the modular focal.

The plane of a modular focal always passes through the directive axis. When the directive axis is the primary, as in the hyperboloid of one sheet, both focals are modular. But in the ellipsoid and the hyperboloid of two sheets, where the primary axis is not directive, only one of the focals can be modular. The plane of an umbilicar focal is

always perpendicular to the directive axis; and therefore, when that axis is the primary, there is no umbilicar focal.*

When the surface is doubly modular, the two moduli m, m' are connected by the relation

$$\frac{\cos^2\phi}{m^2} + \frac{\sin^2\phi}{m'^2} = 1; \tag{16}$$

where ϕ is the angle made by a directive plane with the plane of the focal to which the modulus m belongs. One modulus is greater than unity; the other is less than unity, but greater than the cosine of the angle which the plane of the corresponding focal makes with a directive plane. In the hyperboloid of one sheet, the less modulus is that which belongs to the focal hyperbola. In the cone, the less modulus belongs to the focal lines. Of the two moduli of a cone, that which belongs to the focal lines may be termed the *linear modulus;* and the other, to which only a single focus corresponds, may be called the ***singular modulus***.

§ 6. *Second Class of Surfaces.*—In this class of surfaces, one of the quantities A, B vanishes, or both of them vanish.

I. When $m = \cos\phi$, and ϕ is not zero, A vanishes, but B

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* If the first of the equations (10), when P_1 and Q_1 are both negative, be supposed to express an imaginary focal, there will, in a central surface, be three focals, two modular and one umbilicar; the two modular focals being in the principal planes which pass through the directive axis, and the umbilicar focal in the remaining principal plane. Then, when we know which of the axes is the directive axis, we know which of the three focals is imaginary, because the plane of the imaginary focal is perpendicular to the primary axis. A modular focal may be imaginary, and yet have a real modulus; this occurs in the hyperboloid of two sheets. In the ellipsoid, the imaginary focal has an imaginary modulus. In all cases the two moduli are connected by the relation (16).

It will appear hereafter, that the vertex of the cone is an umbilicar focus. The cone has therefore three focals, none of which is imaginary; but two of them are single points coinciding with the vertex.

does not; and the surface is either a paraboloid or a cylinder.

1°. If the surface is a paraboloid, we may suppose the origin of coordinates to be at its vertex, in which case both H and K vanish, and we have the relations

$$\left.\begin{aligned} G = x_2 - x_1, \qquad & y_1 = y_2 \cos^2\phi, \\ x_2^2 + y_2^2\cos^2\phi - x_1^2 - y_1^2 = 0; & \end{aligned}\right\} \tag{17}$$

the equation of the surface being

$$y^2 \sin^2\phi + z^2 + 2Gx = 0, \tag{18}$$

which shews that the paraboloid is elliptic, having its axis in the direction of x, and the plane of xy for that of its greater principal section. From the relations (17) we obtain the following,

$$\left.\begin{aligned} y_1^2 \tan^2\phi + 2Gx_1 + G^2 = 0, \\ y_2^2 \sin^2\phi\cos^2\phi + 2Gx_2 - G^2 = 0; \end{aligned}\right\} \tag{19}$$

from which we see that the focal and dirigent curves are parabolas, having their axes the same as that of the surface; and their vertices equidistant from the vertex of the surface, but at opposite sides of it. The concavity of each curve is turned in the same direction as that of the section xy. The focus of the focal parabola is the focus of the section xy, and its vertex is the focus of the section xz of the surface; its parameter being the difference of the parameters of these two sections. The parameter of the section xy is a mean proportional between the parameters of the focal and dirigent parabolas.

2°. If the surface is a cylinder, we may make G and H vanish, by taking the origin on its axis. We then have

$$\left.\begin{aligned} x_2 = x_1, \qquad y_1 = y_2\cos^2\phi, \\ K = y_1^2\tan^2\phi = y_2^2\sin^2\phi\cos^2\phi; \end{aligned}\right\} \tag{20}$$

the equation of the cylinder, which is elliptic, being

$$y^2 \sin^2\phi + z^2 = K. \tag{21}$$

Here the focal and dirigent are each a pair of right lines

parallel to the axis of the cylinder, and passing through the foci and directrices of a section perpendicular to the axis. The corresponding focal and dirigent lines lie at the same side of the axis.

II. When $m = 1$, and ϕ is not zero, B vanishes, but A does not.

1°. If the surface is a paraboloid, and the origin of coordinates at its vertex, the quantities G and K vanish; same the equation of the surface becomes

$$x^2 \tan^2 \phi - z^2 = 2\text{H}y, \tag{22}$$

and we have the relations

$$\begin{aligned} \text{K} = y_2 - y_1, \qquad x_1 = x_2 \sec^2 \phi, \\ x_2^2 \sec^2 \phi + y_2^2 - x_1^2 - y_1^2 = 0. \end{aligned} \tag{23}$$

The paraboloid is therefore hyperbolic, its axis being that of y, which is also the directive axis; and as the tangent of ϕ may have any finite value, the plane of xy, which is that of the focal curve, may be either of the principal planes passing through the axis of the surface. The relations (23) give

$$\begin{aligned} x_1^2 \sin^2 \phi - 2\text{H}y_1 - \text{H}^2 = 0, \\ x_2^2 \tan^2 \phi \sec^2 \phi - 2\text{H}y_2 + \text{H}^2 = 0, \end{aligned} \tag{24}$$

for the equations of the focal and dirigent, which are therefore parabolas, having their axes the same as those of the surface, and their concavities turned in the same direction as that of the section xy; their vertices being equidistant from the vertex of the surface, and at opposite sides of it. The focus of the focal parabola is the focus of the section xy, and its vertex is the focus of the section yz, its parameter being the sum of the parameters of these two sections. The parameter of the section xy is a mean proportional between the parameters of the focal and dirigent parabolas.

2°. If the surface is a cylinder, and the origin on its axis, G and H vanish, and we have

$$\begin{gathered} x_1 = x_2 \sec^2 \phi, \qquad y_1 = y_2, \\ - \text{K} = x_1^2 \sin^2 \phi = x_2^2 \tan^2 \phi \sec^2 \phi; \end{gathered} \tag{25}$$

the equation of the cylinder, which is hyperbolic, being

$$x^2 \tan^2 \phi - z^2 = - \text{K}. \tag{26}$$

The focal and dirigent are each a pair of right lines parallel to the axis of the cylinder; the corresponding lines passing through a focus and the adjacent directrix of any section perpendicular to the axis. The directive planes are parallel to the asymptotic planes of the cylinder.

In this case, if $\text{K} = 0$, the surface is reduced to two directive planes, and the focal and dirigent to the intersection of these planes.

III. When $m = 1$, and $\phi = 0$, both A and B vanish, and the surface is the parabolic cylinder. If, as is allowable, we suppose G and K to vanish, the equation of the cylinder becomes

$$z^2 + 2\text{H}y = 0, \tag{27}$$

and we have

$$\begin{gathered} \text{H} = y_2 - y_1, \qquad x_1 = x_2, \\ x_2^2 + y_2^2 - x_1^2 - y_1^2 = 0; \end{gathered} \tag{28}$$

whence

$$y_1 = - \tfrac{1}{2}\text{H}, \qquad y_2 = \tfrac{1}{2}\text{H}. \tag{29}$$

The focal and dirigent are each a right line parallel to the axis of x, the former passing through the focus, the latter meeting the directrix of the parabolic section made by the plane of yz. The plane of xy is the directive plane.

§ 7. We learn from this discussion, that, among the surfaces of the second class, the hyperbolic paraboloid is the only one which admits a twofold modular generation; the modulus, however, being the same for both its focals. In the elliptic paraboloid the modular focal is restricted to the plane of that principal section which has the greater parameter; we shall therefore suppose a parabola to be described in the plane of the other principal section, according to the

law of the modular focals; the law being, that the focus of the parabola shall be the focus of the principal section in the plane of which the parabola lies, and its vertex the focus of the principal section in the perpendicular plane. The parabola so described will have its concavity opposed to that of the surface; it will cut the surface in the umbilics, and will be its umbilicar focal, the only such focal to be found among the surfaces of the second class. We shall of course suppose further, that this focal has a dirigent parabola connected with it by the same law as in the other cases, the vertices of the focal and dirigent being equidistant from that of the surface and at opposite sides of it, while the parameter of the dirigent is a third proportional to the parameters of the focal and of the principal section in the plane of which the curves lie. The two focals of a paraboloid are so related, that the focus of the one is the vertex of the other. The cylinders have no other focals than those which occur above.

§ 8. In this, as in the first class of surfaces, the right line FΔ, joining a focus F with the foot of its corresponding directrix, is perpendicular to the focal line; and the focal and dirigent are reciprocal polars with respect to the section xy of the surface. These properties are easily inferred from the preceding results; but, as they are general, it may be well to prove them generally for both classes of surfaces. Supposing, therefore, the origin of coordinates to be any where in the plane of xy, and writing the equation of the surface in the form

$$(x-x_1)^2+(y-y_1)^2+z^2=\text{L}(x-x_2)^2+\text{M}(y-y_2)^2, \quad (30)$$

which, when identified with (3), gives the relations

$$\begin{aligned} &\text{A}=1-\text{L}, \qquad \text{B}=1-\text{M}, \\ &\text{G}=\text{L}x_2-x_1, \qquad \text{H}=\text{M}y_2-y_1, \\ &\text{K}=\text{L}x_2^2+\text{M}y_2^2-x_1^2-y_1^2, \end{aligned} \quad (31)$$

we find, by differentiating the values of the constants G, H, and K,

$$\begin{gathered} \mathrm{L}dx_2 = dx_1, \qquad \mathrm{M}dy_2 = dy_1, \\ \mathrm{L}x_2\,dx_2 + \mathrm{M}y_2\,dy_2 - x_1\,dx_1 - y_1\,dy_1 = 0. \end{gathered} \tag{32}$$

Hence we obtain

$$(x_2 - x_1)\,dx_1 + (y_2 - y_1)\,dy_1 = 0\,; \tag{33}$$

an equation which expresses that the right line joining the points F and Δ is perpendicular to the line which is the locus of the point F.

Again, the equation of the section xy of the surface being

$$\mathrm{A}x^2 + \mathrm{B}y^2 + 2\mathrm{G}x + 2\mathrm{H}y = \mathrm{K}, \tag{34}$$

the equation of the right line which is, with respect to this section, the polar of a point Δ whose coordinates are x_2, y_2, is

$$(\mathrm{A}x_2 + \mathrm{G})\,x + (\mathrm{B}y_2 + \mathrm{H})\,y = \mathrm{K} - \mathrm{G}x_2 - \mathrm{H}y_2\,; \tag{35}$$

but the relations (31) give

$$\begin{gathered} \mathrm{A}x_2 + \mathrm{G} = x_2 - x_1, \qquad \mathrm{B}y_2 + \mathrm{H} = y_2 - y_1, \\ \mathrm{K} - \mathrm{G}x_2 - \mathrm{H}y_2 = x_1(x_2 - x_1) + y_1(y_2 - y_1)\,; \end{gathered} \tag{36}$$

and hence the equation (35) becomes

$$(x_2 - x_1)(x - x_1) + (y_2 - y_1)(y - y_1) = 0, \tag{37}$$

which, as is evident from (33), is the equation of a tangent applied to the focal at the point F corresponding to Δ. This shows that the focal and dirigent are reciprocal polars with respect to the section xy, and that in this relation, as well as in the other, the points F and Δ are corresponding points.

Supposing F′ and Δ′ to be two other corresponding points on the focal and dirigent, if tangents applied to the focal at F and F′ intersect each other in T, the point T will be the pole of the right line ΔΔ′ with respect to the section xy, as well as the pole of the right line FF′ with respect to

the focal; and hence if any right line be drawn through T, and if P be the pole of this right line with respect to the section, and N its pole with respect to the focal, the points P and N will be on the right lines $\Delta\Delta'$ and FF′ respectively. Now it is useful to observe that the distances $\Delta\Delta'$ and FF′ are always similarly divided (both of them internally or both of them externally) by the points P and N, so that we have ΔP to Δ'P as FN to F′N. This property may be proved directly by means of the foregoing equations; or it may be regarded as a consequence of the following theorem:—If through a fixed point in the plane of two given conics having the same centre, or of two given parabolas having their axes parallel, any pair of right lines be drawn, and their poles be taken with respect to each curve, the distance between the poles relative to one curve will be in a constant ratio to the distance between the poles relative to the other curve.* In fact, the poles of the right lines TF, TF′, with respect to the focal, are F, F′; and their poles with respect to the section xy are Δ, Δ'; therefore, since the focal and the section xy may be taken for the given curves, and the point T for the fixed point, the ratio of FF′ to $\Delta\Delta'$ is the same as the ratio of FN to ΔP or of F′N to Δ'P, and consequently the distances FF′ and $\Delta\Delta'$ are similarly divided in the points N and P.

§ 9. In the equation (30), considered as equivalent to the equation (1), the constants L and M are both positive; but the properties which have been deduced from the former equation are independent of this circumstance, and

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* There is an analogous theorem for two surfaces of the second order which have the same centre, or two paraboloids which have their axes parallel. If through a fixed right line any two planes be drawn, and their poles be taken with respect to each surface, the distance between the poles relative to the one surface will be in a constant ratio to the distance between the poles relative to the other.

equally subsist when one of these constants is supposed to be negative (for they cannot both be negative). This leads us to inquire what surfaces the equation (30) is capable of representing when the constants L and M have different signs; as also, for a given surface, what lines are traced in the plane of xy by points F and Δ, of which x_1, y_1 and x_2, y_2 are the respective coordinates. After the examples already given, this question is easily discussed, and the result is, that the only surfaces which can be so represented are the ellipsoid, the hyperboloid of two sheets, the cone, and the elliptic paraboloid—that is to say, the umbilicar surfaces together with the cone; and that, for an umbilicar surface, the locus of F is the umbilicar focal, and therefore the locus of Δ is the corresponding dirigent; while for the cone the points F and Δ are unique, coinciding with each other and with the vertex of the cone. A geometrical interpretation of this case is readily found; for as L and M have different signs, the right-hand member of the equation (30), if M be the negative quantity, is the product of two factors of the form

$$f(x - x_2) + g(y - y_2), \quad f(x - x_2) - g(y - y_2),$$

in which f and g are constant; and these factors are evidently proportional to the distances of a point whose coordinates are x, y, z, from two planes whose equations are

$$f(x - x_2) + g(y - y_2) = 0, \quad f(x - x_2) - g(y - y_2) = 0,$$

which planes always pass through a directrix, and are inclined at equal and constant angles to the axis of x or of y. Therefore, if F be the focus which belongs to this directrix, the square of the distance of F from any point of this surface is in a constant ratio to the rectangle under the distances of the latter point from the two planes. And these planes are directive planes; because, if a section parallel to one of them be made in the surface, the distance of any point of the section from the other plane will be proportional to the square

of the distance of the same point from the focus; and, as the locus of a point, whose distance from a given plane is proportional to the square of its distance from a given point, is obviously a sphere, it follows that the section aforesaid is the section of a sphere, and consequently a circle; which shows that the plane to which the section is parallel is a directive plane. Thus,* the square of the distance of any point of

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* In attempting to find a geometrical generation for the surfaces of the second order, one of the first things which I thought of, before I fell upon the modular method, was to try the locus of a point such that the square of its distance from a given point should be in a constant ratio to the rectangle under its distances from two given planes; but when I saw that this locus would not represent all the species of surfaces, I laid aside the discussion of it. Some time since, however, Mr. Salmon, Fellow of Trinity College, was led independently, in studying the modular method, to consider the same locus; and he remarked to me, what I had not previously observed, that it offers a property supplementary, in a certain sense, to the modular property; that when the surface is an ellipsoid, for example, the given point or focus is on the focal hyperbola, which the modular property leaves empty. This remark of Mr. Salmon served to complete the theory of the focals, by indicating a simple geometrical relation between a non-modular focal and any point on the surface to which it belongs.

In a memoir "On a new Method of Generation and Discussion of the Surfaces of the second Order," presented by M. Amyot to the Academy of Sciences of Paris, on the 26th December, 1842, the author investigates this same locus, conceiving it to involve that property in surfaces which is analogous to the property of the focus and directrix in the conic sections; and the importance attached to the discovery of such analogous properties induced M. Cauchy to write a very detailed report on M. Amyot's memoir, accompanied with notes and additions of his own (*Comptes rendus des Séances de l'Académie des Sciences*, tom. xvi. pp. 783-828, 885-890; April, 1843); and also occasioned several discussions, principally between M. Poncelet and M. Chasles, relative to that Memoir (*Comptes rendus*, tom. xvi. pp. 829, 938, 947, 1105, 1110). But the property involved in this locus cannot be said to afford a method of generation of the surfaces of the second order, since it applies only to some of the surfaces, and gives an ambiguous result even where it does apply. It is therefore not at all analogous to the aforesaid general property of the conic sections, and moreover it was not new when M. Amyot

the surface from an umbilicar focus bears a constant ratio to the rectangle under the perpendicular distances of the same

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brought it forward. Mr. Salmon had in fact proposed it for investigation to the students of the University of Dublin, at the ordinary examinations in October, 1842; and it was published, towards the end of that year, in the University Calendar for 1843, some months before the date of M. Cauchy's report, by which the contents of M. Amyot's memoir were first made known. The parallelism of the two given planes to the circular sections of the surface is also stated in the Calendar; but this remarkable relation is not noticed by M. Amyot, nor by M. Cauchy. (See the Examination Papers of the year 1842, p. xlv, quest. 17, 18; in the Calendar for 1843.) It is scarcely necessary to add, that the analogue which M. Amyot and other mathematicians have been seeking for, and which was long felt to be wanting in the theory of surfaces of the second order, is no other than the modular property of these surfaces, which appears to be not yet known abroad. M. Poncelet insists much on the importance of extending the signification of the terms *focus* and *directrix*, so as to make them applicable to surfaces; and he supposes this to have been effected, for the first time, by M. Amyot. These terms however, applied in their true general sense to surfaces, had been in use, several years before, among the mathematical students of Dublin, as may be seen by referring to the Calendar (Examination Papers of the year 1838, p. c; 1839, p. xxxi).

The locus above-mentioned, being co-extensive with the umbilicar property, does not represent any surface which can be generated by the right line, except the cone. To remedy this want of generality, M. Cauchy proposes to consider a surface of the second order as described by a point, the square of whose distance from a given point bears a constant ratio either to the rectangle under its distances from two given planes, or to the sum of the squares of these distances. This enunciation, no doubt, takes in both kinds of focals, and all the species of surfaces; but the additional conception is not of the kind required by the analogy in question, nor has it any of the characters of an elementary principle. For the given planes, according to M. Cauchy's idea, do not stand in any simple or natural relation to the surface; and besides there is no reason why, instead of the sum of the squares of the distances from the given planes, we should not take the sum after multiplying the one square by any given positive number, and the other square by another given positive number; nor is there any reason why we should not take other homogeneous functions of these distances. This conception would therefore be found of little use in geometrical applications; while the modular principle, on the contrary, by employing a simple ratio between two right lines, both of which have a natural connexion

point from two directive planes drawn through the directrix corresponding to that focus; and it is easy to see that this ratio, the square root of which we shall denote by μ, is equal to L − M, or, neglecting signs, to the sum of the numerical values of L and M. Of course, if the distances from the directive planes, instead of being perpendicular, be measured parallel to any fixed right line, the ratio will still be constant, though different. For example, if the fixed right line for each plane be that which joins the corresponding umbilic with either focus of the section xy, the ratio of the square to the rectangle will be the square of the number $m \sec \phi$, where m is the modulus, and ϕ the angle which the primary axis makes with a directive plane.

When this property is applied to the cone, the vertex of which is, as we have seen, to be regarded as an umbilicar focus, having the directive axis for its directrix, it indicates that the product of the sines of the angles which any side of the cone makes with its two directive planes is constant.

It is remarkable that the vertex of the cone affords the only instance of a focal point which is at once modular and umbilicar, as well as the only instance of a focal point which is doubly modular. This union of properties it may be conceived to owe to the circumstance that the cone is the asymptotic limit of the two kinds of hyperboloids. For if a

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with the surface, lends itself with the greatest ease to the reasonings of geometry. Indeed the whole difficulty, in extending the property of the directrix to surfaces of the second order, consisted in the discovery of such a simple ratio inherent in all of them.

The polar relation, between the focal and dirigent curves, with respect to the coincident section of the surface, has been perceived by M. Cauchy; and it is therefore proper to mention, what would not otherwise require any special notice, that this relation was stated in my earliest lectures on the subject. The term *modulus*, which I have used for the first time in the present paper, with reference to surfaces of the second order, has been adopted from M. Cauchy, by whom it is employed, however, in a signification entirely different.

series of hyperboloids have the same asymptotic cone, and their primary axes be indefinitely diminished, they will approach indefinitely to the cone; and, in the limit, the focal ellipse and hyperbola of the hyperboloid of one sheet will pass into the vertex and the focal lines of the cone, thus making the vertex doubly modular, while the focal ellipse of the hyperboloid of two sheets will also be contracted into the vertex, and will make that point umbilicar.

When the two directive planes coincide, and become one directive plane, the umbilicar property is reduced to this, that the distances of any point in the surface from the point F and from the directive plane are in a constant ratio to each other; and therefore the surface becomes one of revolution round an axis passing through F at right angles to that plane, the point F being a focus of the meridional section, or the vertex, if the surface be a cone. When the directive planes are supposed to be parallel, but separated by a finite interval, we get the same class of surfaces of revolution, with the addition of the surface produced by the revolution of an ellipse round its minor axis; the point F being still on the axis of revolution, but not having any fixed relation to the surface.

§ 10. If in the equation (30) we supposed the right-hand member to have an additional term containing the product of the quantities $x - x_2$ and $y - y_2$, with a constant coefficient, all the foregoing conclusions regarding the geometrical meaning of that equation would remain unchanged, because the additional term could always be taken away by assigning proper directions to the axes of x and y. If, after the removal of this term, the coefficients of the squares of the aforesaid quantities were both positive, the locus of F would be a modular focal of the surface expressed by the equation; but if one coefficient were positive and the other negative, the locus of F would be an umbilicar focal. The equation in its more general form is evidently that which we should

obtain for the locus of a point S, such that the square of its distance SF from a given point F should be a given homogeneous function of the second degree of its distances from two given planes ; the plane of xy being drawn through F perpendicular to the intersection of these planes, and x_2, y_2 being the coordinates of any point on this intersection, while x_1, y_1 are the coordinates of F. The point F might be any point on one of the focals of the surface described by S; the intersection of the two planes (supposing them always parallel to fixed planes) being the corresponding directrix.

These considerations may be further generalised, if we remark that the equation of any given surface of the second order may be put under the form

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=\text{L}(x-x_2)^2+\text{M}(y-y_2)^2+\text{N}(z-z_2)^2 \\ +\text{L}'(y-y_2)(z-z_2)+\text{M}'(x-x_2)(z-z_2)+\text{N}'(x-x_2)(y-y_2), \quad (38)$$

where L, M, N, L′, M′, N′ are constants, and x_1, y_1, z_1 are conceived to be the coordinates of a certain point F, and x_2, y_2, z_2 the coordinates of another point Δ. The constants L′, M′, N′ may, if we please, be made to vanish by changing the directions of the axes of coordinates ; and when this is done, the new coordinate planes will be parallel to the principal planes of the surface. Then, by proceeding as before, it may be shown that, without changing the surface, we are at liberty, under certain conditions, to make the points F and Δ move in space. The conditions are expressed geometrically by saying that the two surfaces, upon which these points must be always found, are reciprocal polars with respect to the given surface, the points F and Δ being, in this polar relation, corresponding points ; and that the surface which is the locus of F is a surface of the second order, *confocal* with the given one, it being understood that confocal surfaces are those which have the same focal lines. The surface on which Δ lies is therefore also of the second

order, and the right line ΔF is a normal at F to the surface which is the locus of this point. Moreover, if through the point Δ three or more planes be drawn parallel to fixed planes, and perpendiculars be dropped upon them from any point S whose coordinates are x, y, z, the right-hand member of the equation (38) may be conceived to represent a given homogeneous function of the second degree of these perpendiculars; and the given surface may therefore be regarded as the locus of a point S, such that the square of the distance SF is always equal to that function.

§ 11. In the enumeration of the surfaces capable of being generated by the modular method, we miss the five following varieties, which are contained in the general equation of the second degree, but are excluded from that method of generation by reason of the simplicity of their forms—namely, the sphere, the right cylinder on a circular base, and the three surfaces which may be produced by the revolution of a conic section (not a circle) round its primary axis.* These three surfaces are the prolate spheroid, the hyperboloid of two sheets, and the paraboloid of revolution; and the circumstance that the foci of the generating curves are also foci of the surfaces, renders it easy to investigate their focal properties.† In point of simplicity, the excepted surfaces are to the other surfaces of the second order what the circle is to the other conic sections, the circle being, in like manner, excepted from the curves which can be generated by the analogous method *in plano;* and the geometry of the five excepted surfaces may therefore be regarded as comparatively elementary. These five surfaces

* The case of two parallel planes is also excluded, but it is not here taken into account. The case of two parallel right lines is in like manner excluded from the corresponding generation of lines of the second order.

† A paper by M. Chasles, on these surfaces of revolution, will be found in the Memoirs of the Academy of Brussels, tom. v. (An. 1829).

were, in fact, studied by the Greek geometers,* and, along with the oblate spheroid and the cone, they make up all the surfaces of the second order with which the ancients were acquainted. Except the cone, the surfaces considered by them are all of revolution; and there is only one surface of revolution, the hyperboloid of one sheet, which was not noticed until modern times. This surface is mentioned (under the name of the hyperbolic cylindroid) by Wren,† who remarks that it can be generated by the revolution of a right line round another right line not in the same plane. As to the general conception of surfaces of the second order, the suggestion of it was reserved for the algebraic geometry of Descartes. In that geometry the curves previously known as sections of the cone are all expressed by the general equation of the second degree between two coordinates; and hence it occurred to Euler‡ about a century ago, to examine and classify the different kinds of surfaces comprised in the general equation of the second degree among three coordinates. The new and more general forms thus brought to light have since engaged a large share of the attention of geometers; but the want of some other than an algebraic principle of connexion has prevented any great progress from being made in the investigation of such of their properties as do not immediately depend on transformations of coordinates. This want the modular method of generation perfectly supplies, by evolving the different forms from a simple geometrical conception, at the same time that it brings them within the range of ideas familiar to the ancient geometry, and places their relation to the conic sections in a striking point of view.

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* The hyperboloid of two sheets, and the paraboloid of revolution, were known by the name of *conoids*. Archimedes has left a treatise on Conoids and Spheroids, as well as a treatise on the Sphere and Cylinder.

† In the Philosophical Transactions for the year 1669, p. 961.

‡ See his *Introductio in Analysin Infinitorum*, p. 373; Lausanne, 1748.

It may be well to remark that the excepted surfaces are *limits* of surfaces which can be generated modularly, as the circle is the limit of the ellipse in the analogous generation of the conic sections. Thus the sphere is the limit of an oblate spheroid, one of whose axes remains constant, while its focal circle is indefinitely diminished; and the right circular cylinder is the limit of an elliptic cylinder, whose focal lines are conceived to approach indefinitely to coincidence with each other and with the axis of the cylinder, while one of the axes of the principal elliptic section remains constant. In these cases the dirigent lines, along with the directrices, move off to infinity. The other three excepted surfaces correspond to the supposition $\phi = 90°$, which was excluded in the discussion of the general equation (1). For if we make $m \sec \phi = n$, the quantity which constitutes the right-hand member of that equation may be written

$$n^2 (x - x_2)^2 + n^2 (y - y_2)^2 \cos^2 \phi ;$$

and if we suppose n to remain finite and constant, while ϕ approaches to 90°, and m indefinitely diminishes, this quantity will approach indefinitely to $n^2 (x - x_2)^2$, which will be its limiting value when $\phi = 90°$. But $x - x_2$ is the distance of the point S from a fixed plane intersecting the axis of x perpendicularly at the distance x_2 from the origin of coordinates; and therefore, in the limit, the equation expresses that the distances of any point S of the surface from the focus F and from this fixed plane, are to each other as n to unity, that is, in a constant ratio; which is a common property of the three surfaces in question. This property also belongs to the right cone, but the right cone does not rank among the excepted surfaces.

§ 12. We have seen that, when the modulus is unity, any plane parallel to either of the directive planes intersects the surface in a right line; whence it follows, that through any point on the surface of a hyperbolic paraboloid two right

lines may be drawn which shall lie entirely in the surface. The plane of these right lines is of course the tangent plane at that point, and therefore every tangent plane intersects the surface in two right lines. This is otherwise evident from considering that the sections parallel to a given tangent plane are similar hyperbolas, whose centres are ranged on a diameter passing through the point of contact, and whose asymptotes, having always the same directions, are parallel to two fixed right lines which we may suppose to be drawn through that point. For as the distance between the plane of section and the tangent plane diminishes, the axes of the hyperbola diminish; and they vanish when that dis-stance vanishes, the hyperbola being then reduced to its asymptotes. The tangent plane therefore intersects the surface in the two fixed right lines aforesaid. The same reasoning, it is manifest, will apply to any other surface of the second order, which has hyperbolic sections parallel to its tangent planes; and therefore the hyperboloid of one sheet, which is the only other such surface,* is also intersected in two right lines by any of its tangent planes. These right lines are usually called the *generatrices* of the surface.

From what has been said, it appears that the generatrices of the hyperbolic paraboloid, and the asymptotes of its sections (all its sections, except those made by planes parallel to the axis, being hyperbolas), are parallel to the directive planes. The generatrices of the hyperboloid of one sheet, and the asymptotes of its hyperbolic sections, are parallel to the sides of the asymptotic cone; because any section of the

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* The double generation of these two surfaces by the motion of a right line has been long known. It appears to have been discovered and fully discussed by some of the first pupils of the Polytechnic School of Paris. This mode of generation had, however, been remarked by Wren, with regard to the hypreboloid of revolution. It does not seem to have been observed, that the existence of rectilinear generatrices is included in the idea of hyperbolic sections parallel to a tangent plane.

hyperboloid is similar to a parallel section of the asymptotic cone, and when the latter section is a hyperbola its asymptotes are parallel to two sides of the cone.

PART II.—PROPERTIES OF SURFACES OF THE SECOND ORDER.

§ 1. In the preceding part of this paper it has been necessary to enter into details for the purpose of communicating fundamental notions clearly. In the following part, which will contain certain properties of surfaces of the second order, we shall be as brief as possible; giving demonstrations of the more elementary theorems, but confining ourselves to a short statement of the rest.

Many consequences follow from the principles already laid down.

Through any directrix of a surface of the second order let a fixed plane be drawn cutting the surface, and let S be any point of the section. If the directrix and its focus F be modular, and if a plane always parallel to the same directive plane be conceived to pass through S and to cut the directrix in D, the directive distance SD will be always parallel to a given right line, and will therefore be in a constant ratio to the perpendicular distance of S from the directrix. This perpendicular distance will consequently bear a given ratio to SF, the distance of the point S from the focus. And the same thing will be true when the directrix and focus are umbilicar, because the perpendicular distance of the point S from the directrix will be in a constant ratio to its distance from each directive plane drawn through the directrix.

The fixed plane of section will in general contain another directrix parallel to the former, and belonging to the same focal; and it is evident that the perpendicular distance of S from this other directrix will be in a given ratio to its distance SF′ from the corresponding focus F′, the ratio being the same as in the former case. Hence, according as the point S lies between the two directrices, or at the same side

of both, the sum or difference of the distances SF and SF′ will be constant.

If the plane of section pass through either of the foci, as F, this focus and its directrix will manifestly be the focus and directrix of the section. In this case the plane of section will be perpendicular to the focal at F. And if the surface be a cone, the point F being anywhere on one of its focal lines, the distance of the point S from the directrix will be in a constant ratio to its perpendicular distance from the dirigent plane which contains the directrix, and therefore this perpendicular distance will be in a given ratio to the distance SF. Now calling V the vertex of the cone, and taking SV for radius, the perpendicular distance aforesaid is the sine of the angle which the side SV of the cone makes with the dirigent plane; and SF, which is perpendicular to VF, is the sine of the angle SVF. Consequently the sines of the angles which any side of a cone makes with a dirigent plane and the corresponding focal line are in a given ratio to each other.

§ 2. Conceive a surface of the second order to be intersected in two points S, S′ by a right line which cuts two parallel directrices in the points E, E′, and let F, F′ be the foci corresponding respectively to these directrices. The perpendicular distances of the points S, S′ from the first directrix and from the second are to each other as the lengths SE, S′E, SE′, S′E′ respectively, and therefore the ratios of FS to SE, of FS′ to S′E, of F′S to SE′, and of F′S′ to S′E′ are all equal.

Hence, the right line FE bisects one of the angles made by the right lines FS and FS′; and the right line F′E′ bisects one of the angles made by F′S and F′S′.

When the points S, S′ are at the same side of E, the angle supplemental to SFS′ is that which is bisected by the right line FE. Now if the point S be fixed, and S′ approach to it indefinitely, the angle SFE will approach inde-

finitely to a right angle. Therefore if a right line touching the surface meet a directrix in a certain point, the distance between this point and the point of contact will subtend a right angle at the focus which corresponds to the directrix. And if a cone circumscribing the surface have its vertex in a directrix, the curve of contact will be in a plane drawn through the corresponding focus at right angles to the right line which joins that focus with the vertex.

When the surface intersected by the right line SS′ is a cone, suppose this line to lie in the plane of the focus F and its directrix, that is, in the plane which is perpendicular at F to the focal line VF (the vertex of the cone being denoted, as before, by V); the angles made by the right lines FE, FS, FS′, are then the same as the angles made by planes drawn through VF and each of the right lines VE, VS, VS′; and the last three right lines are the intersections of a plane VSS′ with the dirigent plane on which the point E lies, and with the surface of the cone. Therefore if a plane passing through the vertex of a cone intersect its surface in two right lines, and one of its dirigent planes in another right line, and if a plane be drawn through each of these right lines respectively and the focal line which belongs to the dirigent plane, the last of the three planes so drawn will bisect one of the angles made by the other two. And hence, if a plane touching a cone along one of its sides intersect a dirigent plane in a certain right line, and if through this right line and the side of contact two planes be drawn intersecting each other in the focal line which corresponds to the dirigent plane, the two planes so drawn will be at right angles to each other.

Let a right line touching a surface of the second order in S meet two parallel directrices in the points E, E′, and let F, F′ be the corresponding foci. Then the triangles FSE and F′SE′ are similar, because the angles at F and F′ are right angles, and the ratio of FS to SE is the same as the ratio of F′S

to SE′. Therefore the tangent EE′ makes equal angles with the right lines drawn from the point of contact S to the foci F, F′. When the surface is a cone, let the tangent be perpendicular to the side VS which passes through the point of contact; the angles FSE and F′SE′ are then the angles which the tangent plane VEE′ makes with the planes VSF and VSF′, because the right line FE is perpendicular to the plane VSF, and the right line F′E′ is perpendicular to the plane VSF′. Therefore the tangent plane of a cone makes equal angles with the planes drawn through the side of contact and each of the focal lines.

Supposing a section to be made in a surface of the second order by a plane which cuts any directrix in the point E, if the focus F belonging to this directrix be the vertex of a cone having the section for its base, the right line FE will be an axis of the cone. For if through FE any plane be drawn cutting the base of the cone in the points S, S′, one of the angles made by the sides FS, FS′ which pass through these points will always be bisected by the right line FE; and this is the characteristic property of an axis.

§ 3. Two surfaces of the second order being supposed to have the same focus, directrix, and directive planes, so that they differ only in the value of the modulus m, or of the umbilicar ratio μ (see Part I. § 9), let a right line passing through any point E of the directrix cut one surface in the points S, S′, and the other in the points S_0, S_1, and conceive right lines to be drawn from all these points to the common focus F. Since, if ratios be expressed by numbers, the ratio of FS to SE (or of FS′ to S′E) is to the ratio of FS_0 to S_0E (or of FS_1 to S_1E) as the value of m for the one surface is to its value for the other, when the focus is modular, or as the value of $\bar{\mu}$ for the one surface is to its value for the other when the focus is umbilicar, the sines of the angles EFS_0 and EFS (or of the angles EFS_1 and EFS′) are in a constant proportion to each other, because these sines are pro-

portional to those ratios. And since the right line FE bisects the angles SFS′ and S_0FS_1, both internally or both externally, in which case the angles SFS_0 and $S'FS_1$ are equal, or else one internally and the other externally in which case the angles SFS_0 and $S'FS_1$ are supplemental, it is easy to infer, from the constant ratio of the aforesaid sines, that in the first case the product, in the second case the ratio of the tangents of the halves of the angles SFS_0 and $S'FS_0$ (or of the halves of the angles SFS_1 and $S'FS_1$) is a constant quantity.

If the point S′ approximate indefinitely to S, the right line passing through these points will approach indefinitely to a tangent. Therefore when two surfaces are related as above, if a right line passing through any point E of their common directrix intersect one surface in the points S_0, S_1, and touch the other in the point S, the chord S_0S_1 will subtend a constant angle at the common focus F, and this angle will be bisected, either internally or externally, by the right line FS drawn from the focus to the point of contact. And the angle EFS being then a right angle, the cosine of the angle SFS_0 or SFS_1 will be equal to the ratio of the less value of m or μ to the greater.*

§ 4. Among the surfaces of the second order the only one which has a point upon itself for a modular focus is the cone, the vertex of which is such a focus, related either to the internal or to the mean axis as directrix. In the latter relation the vertex belongs to the series of foci which are ranged on the focal lines. To see the consequence of this, let V be the vertex of the cone, and VW its mean axis perpendicular to the plane of the focal lines. On one of the focal lines and its dirigent assume any corresponding

* See Exam. Papers, An. 1839, p. xxxi. questions 9, 10. These and some of the preceding theorems were originally stated with reference to modular foci only. They are now extended to umbilicar foci.

points F and Δ, and let ΔD be the directrix passing through Δ. Then if a directive plane, drawn through any point S of the surface, cut this directrix in D and the mean axis in W, the ratio of SF to SD will be expressed by the linear modulus, as will also the ratio of VF to WD, since V is a point of the surface, and WD is equal to the directive distance of V from ΔD. But since V is a focus to which the mean axis is directrix, the ratio of SV to SW is expressed by the same modulus. Thus the triangles SVF and SWD are similar, the sides of the one being proportional to those of the other. Therefore the angle SVF is equal to the angle SWD; that is to say, the angle which the side VS of the cone makes with the focal line VF is equal to the angle contained by two right lines WD and WS, of which one is the intersection of the directive plane with the dirigent plane VWD corresponding to VF, and the other is the intersection of the directive plane with the plane VWS passing through the mean axis and the side VS of the cone.

Hence it appears that the sum of the angles (properly reckoned) which any side of the cone makes with its two focal lines is constant. For if F′ be a point on the other focal line, and D′ the point where the directrix corresponding to F′ is intersected by the same directive plane SWD, it may be shown as above that the angle SVF′ is equal to the angle SWD′, that is, to the angle made by the right line WS with the right line WD′ in which the directive plane intersects the dirigent plane corresponding to VF′. Conceiving therefore the points F, F′, S, and with them the points D, D′, to lie all on the same side of the principal plane which is perpendicular to the internal axis, the right line WS will lie between the right lines WD and WD′, and the sum of the angles SVF and SVF′ will be equal to the angle DWD′, which is a constant angle, being contained by the right lines in which a directive plane intersects the two dirigent planes of the cone. This constant angle will be

found to be equal, as it ought to be, to one of the angles made by the two sides of the cone which are in the plane of the focal lines, namely to the angle within which the internal axis lies.

If we conceive the cone to have its vertex at the centre of a sphere, and the points F, F′, S to be on the surface of this sphere, the arcs of great circles connecting the point S with each of the fixed points F, F′ will have a constant sum. The curve formed by the intersection of the sphere and the cone may therefore, from analogy, be called a *spherical ellipse,* or, more generally, a *spherical conic,* because, by removing one of its foci F, F′ to the opposite extremity of the diameter of the sphere, the difference of the arcs SF and SF′ will be constant, which shows that the spherical curve is analogous to the hyperbola as well as to the ellipse. Either of these plane curves may, in fact, be obtained as a limit of the spherical curve when the sphere is indefinitely enlarged, according as the diameter along which the enlargement takes place, and of which one extremity may be conceived to be fixed while the other recedes indefinitely, coincides with the internal or with the directive axis of the cone. The fixed extremity becomes the centre of the limiting curve, which is an ellipse in the first case, and a hyperbola in the second.

The great circle touching a spherical conic at any point makes equal angles with the two arcs of great circles which join that point with the foci, because the sum of these arcs is constant. This is identical with a property already demonstrated relative to the tangent planes of the cone. Indeed it is obvious that the properties of the cone may also be stated as properties of the spherical conic, and this is frequently the more convenient way of stating them.

§ 5. If the sides of one cone be perpendicular to the tangent planes of another, the tangent planes of the former will be perpendicular to the sides of the latter. For the plane of two sides of the first cone is perpendicu-

lar to the intersection of the two corresponding tangent planes of the second cone; and as these two sides approach indefinitely to each other, their plane approaches to a tangent plane, while the intersection of the two corresponding tangent planes of the second cone approaches indefinitely to a side of the cone. Thus any given side of the one cone corresponds to a certain side of the other; and any side of either cone is perpendicular to the plane which touches the other along the corresponding side. This reasoning applies to cones of any kind.

Two cones so related may be called *reciprocal cones*. When one is of the second order, it will be found that the other is also of the second order, and that, in their equations relative to their axes, which are obviously parallel or coincident, the coefficients of the squares of the corresponding variables are reciprocally proportional, so that the equations

$$\mathrm{P}x^2 + \mathrm{Q}y^2 + \mathrm{R}z^2 = 0, \qquad \frac{x^2}{\mathrm{P}} + \frac{y^2}{\mathrm{Q}} + \frac{z^2}{\mathrm{R}} = 0, \qquad (1)$$

express two such cones which have a common vertex. These cones have the same internal axis, but the directive axis of the one coincides with the mean axis of the other, and it may be shown from the equations that the directive planes of the one are perpendicular to the focal lines of the other. The two curves in which these cones are intersected by a sphere, having its centre at their common vertex, are reciprocal spherical conics. In general, two curves traced on the surface of a sphere may be said to be reciprocal to each other, when the cones passing through them, and having a common vertex at the centre of the sphere, are reciprocal cones. Any given point of the one curve corresponds to a certain point of the other, and the great circle which touches either curve at any point is distant by a quadrant from the corresponding point of the other curve.

By means of these relations any property of a cone of the second order, or of a spherical conic, may be made to produce

a reciprocal property. Thus, we have seen that the tangent plane of a cone makes equal angles with two planes passing through the side of contact and through each of the focal lines; therefore, drawing right lines perpendicular to the planes, and planes perpendicular to the right lines here mentioned, we have, in the reciprocal cone, a side making equal angles with the right lines in which the directive planes of this cone are intersected by a plane touching it along that side. It is therefore a property of the cone, that the intersections of a tangent plane with the two directive planes make equal angles with the side of contact; a property which it is easy to prove without the aid of the reciprocal cone.

The two directive sections drawn through any point S of a given surface of the second order may, when they are circles, be made the directive sections of a cone, and this may obviously be done in two ways. Each of the two cones so determined will be touched by the plane which touches the given surface at the point S, because the right lines which are tangents to the two circular sections at that point, are tangents to each cone as well as to the given surface; therefore the side of contact of each cone bisects one of the angles made by these two tangents; and hence the two sides of contact are the principal directions in the tangent plane at the point S, that is, they are the directions of the greatest and least curvature of the given surface at that point; for these directions are parallel to the axes of a section made in the surface by a plane parallel to the tangent plane, and the axes of any section bisect the angles contained by the right lines in which the plane of section cuts the two directive planes.

§ 6. It has been shown that the sum of the angles which any side of a cone makes with its focal lines is constant. Hence we obtain the reciprocal property, that* the sum of

* This property, and that to which it is reciprocal, as well as some other properties of the cone, were, together with the idea of reciprocal cones and of

the angles (properly reckoned) which any tangent plane of a cone makes with its two directive planes is constant. This property may be otherwise proved as follows.

Through a point assumed anywhere in the side of contact, let two directive planes be drawn. As the circles in which the cone is cut by these planes have a common chord, they are circles of the same sphere; and a tangent plane applied to this sphere, at the aforesaid point, coincides with the tangent plane of the cone, because each *tangent* plane contains the tangents drawn to the two circles at that point. The common chord of the circles is bisected at right angles by the principal plane which is perpendicular to the directive axis, and therefore that principal plane contains the centres of the two circles and the centre of the sphere. Now the acute angle made by a tangent plane of a sphere with the plane of any small circle passing through the point of contact, is evidently half the angle subtended at the centre of the sphere by a diameter of that circle; therefore the acute angles, which the common tangent plane of the cone and of the sphere above-mentioned makes with the planes of the directive sections, are the halves of the angles subtended at the centre of the sphere by the diameters of the sections. But the diameters which lie in the principal plane already spoken of, and are terminated by two sides of the cone, are chords of the great circle in which that plane intersects the

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spherical conics, suggested by my earliest researches connected with the mechanical theory of rotation and the laws of double refraction. I was not then aware that the focal lines of the cone had been previously discovered, nor that the spherical conic had been introduced into geometry. Indeed all the properties of the cone which are given in this paper were first presented to me in my own investigations. Its double modular property, related to the vertex as focus, was one of the propositions in the theory of the rotation of a solid body, and was used in finding the position of the axis of rotation within the body at a given time. But the modular property common to all the surfaces of the second order was not discovered until some years later.

sphere; and the halves of the angles which they subtend at its centre are equal to the angles in the greater segments of which they are the chords, and consequently equal to the two adjacent acute angles of the quadrilateral which has these chords for its diagonals. Hence, as two opposite angles of the quadrilateral are together equal to two right angles, it follows that the four angles of the quadrilateral represent the four angles, the obtuse as well as the acute angles, which the tangent plane of the cone makes with the planes of the directive sections; the two angles of the quadrilateral which lie opposite to the same diagonal being equal to the acute and obtuse angles made by the tangent plane with the plane of the section of which that diagonal is the diameter.

Thus any two adjacent angles of the quadrilateral may be taken for the angles which the tangent plane of the cone makes with the directive planes. If we take the two adjacent angles which lie in the same triangle with the angle κ contained by the two sides of the cone that help to form the quadrilateral, the sum of these two angles will be equal to two right angles diminished by κ; and if we take the two remaining angles of the quadrilateral, their sum will be equal to two right angles increased by κ; both which sums are constant. But if we take either of the other pairs of adjacent angles, the difference of the pair will be constant, and equal to κ.

The same conclusion may be deduced as a property of the spherical conic. Let a great circle touching this curve be intersected in two points, one on each side of the point of contact, by the two directive circles, that is, by two great circles whose planes are directive planes of the cone which passes through the conic and has its vertex at the centre of the sphere. Since the right lines in which the tangent plane of a cone intersects the directive planes are equally inclined to the side of contact, the arc intercepted between the points where the tangent circle of the conic intersects the directive

circles is bisected in the point of contact; therefore, either of the spherical triangles whose base is the tangent arc so intercepted, and whose other two sides are the directive circles, has a constant area; because, if we suppose the tangent arc to change its position through an indefinitely small angle, and to be always terminated by the directive circles, the two little triangles bounded by its two positions and by the two indefinitely small directive arcs which lie between these positions, will have their nascent ratio one of equality, so that the area of either of the spherical triangles mentioned above, will not be changed by the change in the position of its base. But in each of these triangles the angle opposite the base is constant; therefore the sum of the angles at the base is constant.

From this reasoning it appears that if a spherical triangle have a given area, and two of its sides be fixed, the third side will always touch a spherical conic having the fixed sides for its directive arcs, and will be always bisected in the point of contact.

§ 7. The intersection of any given central surface of the second order with a concentric sphere is a spherical conic, since the cone which passes through the curve of intersection and has its vertex at the common centre, is of the second order. The cylinder also, which passes through the same curve and has its side parallel to any of the arcs of the given surface, is of the second order; and the cone, the cylinder, and the given surface are *condirective,* that is, the directive planes of one of them are also the directive planes of each of the other two. This may be seen from the equations of the different surfaces; for, in general, two surfaces, whose principal planes are parallel, will be condirective, if, when their equations are expressed by coordinates perpendicular to these planes, the differences of the coefficients of the squares of the variables in the equation of the one be pro-

portional to the corresponding differences in the equation of the other.

If any given surface of the second order be intersected by a sphere whose centre is any point in one of the principal planes, the cylinder passing through the curve of intersection, and having its side perpendicular to that principal plane, will be of the second order, and will be condirective with the given surface. This cylinder, when its side is parallel to the directive axis, is hyperbolic; otherwise it is elliptic. If a paraboloid be cut by any plane, the cylinder which passes through the curve of section and has its side parallel to the axis of the paraboloid, will be condirective with that surface; and it will be elliptic or hyperbolic, according as the paraboloid is elliptic or hyperbolic.*

If two concentric surfaces of the second order be reciprocal polars with respect to a concentric sphere, the directive axis of the one surface will coincide with the mean axis of the other, and the directive planes of the one will be perpendicular to the asymptotes of the focal hyperbola of the other. When one of the surfaces is a hyperboloid, the other is a hyperboloid of the same kind; the asymptotes of the focal hyperbola of each surface are the focal lines of its asymptotic cone; and the two asymptotic cones are reciprocal.

When any number of central surfaces of the second order are confocal, or, more generally, when their focal hyperbolas have the same asymptotes, it is obvious that their reciprocal surfaces, taken with respect to any sphere concentric with them, are all condirective.

§ 8. If a diameter of constant length, revolving within a

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* I have introduced the terms *directive* and *condirective*, as more general than the terms *cyclic* and *biconcyclic* employed by M. Chasles. The latter terms suggest the idea of circular sections, and therefore could not properly be used with reference to the hyperbolic paraboloid, or to the hyperbolic or parabolic cylinder, in each of which surfaces a directive section is a right line.

given central surface, describe a cone having its vertex at the centre, the extremities of the diameter will lie in a spherical conic. And if the cone be touched by any plane, the side of contact will evidently be normal to the section which that plane makes in the given surface, and will therefore be an axis of the section. As the axes of a section always bisect the angles made by the two right lines in which its plane intersects the directive planes of the surface, and as the cone aforesaid has the same directive planes with the given surface, it follows that the right lines in which a tangent plane of a cone cuts its directive planes are equally inclined to the side of contact; a theorem which has been already obtained in another way.

If a section be made in a given central surface by any plane passing through the centre, the cone described by a constant semidiameter equal to either semiaxis of the section will touch the plane of section; for if it could cut that plane, a semiaxis would be equal to another radius of the section. Denoting by r, r' the semiaxes of the section, conceive two cones to be described by the revolution of two constant semidiameters equal to r and r' respectively. These cones are condirective with the given surface, and have the plane of section for their common tangent plane. Supposing that surface to be expressed by the equation

$$\frac{x^2}{\text{P}} + \frac{y^2}{\text{Q}} + \frac{z^2}{\text{R}} = 1, \tag{2}$$

and the directive axis to be that of y, the axis of x will be the internal axis of one cone, say of that described by r, and the axis of z will be the internal axis of the other cone. Let κ be the angle made by the two sides of the first cone which lie in the plane xz, and κ' the angle made by the two sides of the second cone which lie in the same plane; the former angle being taken so as to contain the axis of x within it, and the latter so as to contain within it the axis of z.

Then, considering r, r' as radii of the section xz of the surface, we have obviously

$$\frac{1}{r^2} = \frac{\cos^2 \frac{1}{2}\kappa}{\text{P}} + \frac{\sin^2 \frac{1}{2}\kappa}{\text{R}} = \tfrac{1}{2}\left(\frac{1}{\text{P}} + \frac{1}{\text{R}}\right) + \tfrac{1}{2}\left(\frac{1}{\text{P}} - \frac{1}{\text{R}}\right)\cos\kappa,$$
$$\frac{1}{r'^2} = \frac{\cos^2 \frac{1}{2}\kappa'}{\text{R}} + \frac{\sin^2 \frac{1}{2}\kappa'}{\text{P}} = \tfrac{1}{2}\left(\frac{1}{\text{P}} + \frac{1}{\text{R}}\right) - \tfrac{1}{2}\left(\frac{1}{\text{P}} - \frac{1}{\text{R}}\right)\cos\kappa'; \tag{3}$$

observing that when these formulæ give a negative value for r^2 or r'^2, in which case the surface expressed by the equation (2) must be a hyperboloid, the direction of r or r' meets, not that surface, but the surface of the conjugate hyperboloid expressed by the equation

$$\frac{x^2}{\text{P}} + \frac{y^2}{\text{Q}} + \frac{z^2}{\text{R}} = -1. \tag{4}$$

Now calling θ and θ' the angles made by the tangent plane of the cones with the directive planes of the given surface, which are also the directive planes of each cone, the angles κ, κ' depend on the sum or difference of θ and θ'. If the latter angles be taken so that their sum may be equal to the supplement of κ, their difference will be equal to κ', and the formulæ (3) will become

$$\frac{1}{r^2} = \tfrac{1}{2}\left(\frac{1}{\text{P}} + \frac{1}{\text{R}}\right) - \tfrac{1}{2}\left(\frac{1}{\text{P}} - \frac{1}{\text{R}}\right)\cos(\theta + \theta')$$
$$\frac{1}{r'^2} = \tfrac{1}{2}\left(\frac{1}{\text{P}} + \frac{1}{\text{R}}\right) - \tfrac{1}{2}\left(\frac{1}{\text{P}} - \frac{1}{\text{R}}\right)\cos(\theta - \theta'), \tag{5}$$

by which the semiaxes of any central section are expressed in terms of the non-directive semiaxes of the surface, and of the angles which the plane of section makes with the directive planes.*

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* See the Transactions of the Royal Irish Academy, vol. xxi., as before cited. The formulæ (5) were first given, for the case of the ellipsoid, by Fresnel, in his Theory of Double Refraction, Mémoires de l'Institut, tom. vii., p. 155.

§ 9. From the centre O of the surface expressed by equation (*2*) let a right line $O\Sigma$ be drawn cutting perpendicularly in Σ the plane which touches the surface at S. Let σ denote the length of the perpendicular $O\Sigma$, and α, β, γ the angles which it makes with x, y, z. Then

$$\sigma^2 = \text{P} \cos^2\alpha + \text{Q} \cos^2\beta + \text{R} \cos^2\gamma. \tag{6}$$

From this formula it is manifest, that if three planes touching the surface be at right angles to each other, the sum of the squares of their perpendicular distances from the centre will be equal to the constant quantity $\text{P} + \text{Q} + \text{R}$, and therefore the point of intersection of the planes will lie in the surface of a given sphere. If another surface represented by the equation

$$\frac{x^2}{\text{P}_0} + \frac{y^2}{\text{Q}_0} + \frac{z^2}{\text{R}_0} = 1,$$

be touched by a plane cutting $O\Sigma$ perpendicularly in Σ_0, and if σ_0 be the length of $O\Sigma_0$, then

$$\sigma_0^2 = \text{P}_0 \cos^2\alpha + \text{Q}_0 \cos^2\beta + \text{R}_0 \cos^2\gamma;$$

and therefore when the two surfaces are confocal, that is, when

$$\text{P} - \text{P}_0 = \text{Q} - \text{Q}_0 = \text{R} - \text{R}_0 = k,$$

we have $\sigma^2 - \sigma_0^2 = k$, which is a constant quantity. Hence if three confocal surfaces be touched by three rectangular planes, the sum of the squares of the perpendiculars dropped on these planes from the centre will be constant, and the locus of the intersection of the planes will be a sphere.

The focal curves of a given surface are the limits of surfaces confocal with it,* when these surfaces are conceived,

* It was by this consideration, arising out of the theorems given in this and the next section about confocal surfaces, that I was led to perceive the nature of the focal curves, and the analogy between their points and the foci of

by the progressive diminution of their mean or secondary axes, to become flattened, and to approach more and more nearly to a plane passing through the primary axis. And it will appear hereafter, that if a *bifocal right line*, that is, a right line passing through both focal curves, be the intersection of two planes touching these curves, those two planes will be at right angles to each other. Therefore the locus of the point where a tangent plane of a given central surface is intersected perpendicularly by a bifocal right line is a sphere. The primary axis of the surface is evidently the diameter of this sphere.

Hence we conclude that the locus of the point where a tangent plane of a paraboloid is intersected perpendicularly by a bifocal right line is a plane touching the paraboloid at its vertex. For a paraboloid is the limit of a central surface whose primary axis is prolonged indefinitely in one direction, and a plane is the corresponding limit of the sphere described on that axis as diameter. As this consideration is frequently of use in deducing properties of paraboloids from those of central surfaces, it may be well to state it more particularly. It is to be observed, then, that the indefinite extension of the primary axis at one extremity may take place according to any law which leaves the other extremity always at a finite distance from a given point, and gives a finite limiting parameter to each of the principal sections of the surface which pass through that axis. The simplest supposition is, that one extremity of the axis and the adjacent foci of those two principal sections remain fixed, while the other extremity and the other foci move off, with the centre, to distances which are conceived to increase without limit. Then, at any finite distances from the fixed

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conics. And I regarded that analogy as fully established when I found (in March or April, 1832) that the normal at any point of a surface of the second order is an axis of the cone which has that point for its vertex and a focal for its base.

points, the focal curves approach indefinitely to parabolas, as do also all sections of the surface which pass through the primary axis, while the surface itself approaches indefinitely to a paraboloid; so that the limit of the central surface is a paraboloid having parabolas for its focal curves. The limit of an ellipsoid, or of a hyperboloid of two sheets, is an elliptic paraboloid, having one of its focals modular and the other umbilicar, like each of the central surfaces from which it may be derived; and the limit of a hyperboloid of one sheet is a hyperbolic paraboloid, having, like that hyberboloid, both its focals modular.

§ 10. Let the plane touching at S the surface expressed by equation (2), intersect the axis of x in the point X, and let the normal applied at S intersect the planes yz, xz, xy, in the points L, M, N respectively. Since the section made in the surface by a plane passing through OX and the point S has one of its axes in the direction of OX, it appears, by an elementary property of conics, that the rectangle under OX and the coordinate x of the point S is equal to the quantity P; but that coordinate is to LS as OΣ or σ is to OX, and therefore the rectangle under σ and LS is equal to P. Similarly the rectangle under σ and MS is equal to Q, and the rectangle under σ and NS is equal to R. Thus the parts of the normal intercepted between the point S and each of the principal planes, are to each other as the squares of the semiaxes respectively perpendicular to these planes; the square of an imaginary semiaxis being regarded as negative, and the corresponding intercept being measured from S in a direction opposite to that which corresponds to a real semiaxis.

The rectangle under σ and the part of the normal intercepted between two principal planes, is equal to the difference of the squares of the semiaxes which are perpendicular to these planes. This rectangle is therefore constant, not only

for a given surface, but for all surfaces which are confocal with it.

Hence the part of the normal intercepted between two principal planes bears a given ratio to the part of it intercepted between one of these and the third principal plane, whether the normal be applied at any point of a given surface, or at any point of a surface confocal with it.

If therefore normals to a series of confocal surfaces be all parallel to a given right line, they must all lie in the same plane passing through the common centre of the surfaces, because otherwise the parts of any such normal, which are intercepted between each pair of principal planes, would not be in a constant ratio to each other.

The point S being the point at which any of these parallel normals is applied, the plane touching the surface at S is parallel to a given plane, the perpendicular OΣ dropped upon it from the centre has a given direction, the plane OSΣ is fixed, and the directions of the lines OL, OM, ON in which this plane intersects the principal planes are also fixed. And as the angle OΣS is always a right angle, and the normal at S is always parallel to OΣ, the distance SΣ bears a given ratio to each of the distances OL, OM, ON, and therefore also to each of the intercepts MN, LN, LM. Hence, since the rectangle under OΣ and any one of these intercepts is constant, the rectangle under OΣ and SΣ is constant.

Therefore if a series of confocal central surfaces be touched by parallel planes, the points of contact will all lie in one plane, and their locus, in that plane, will be an equilateral hyperbola, having its centre at the centre of the surfaces, and having one of its asymptotes perpendicular to the tangent planes. This hyperbola evidently passes through two points on each of the focal curves, namely the points where the tangent to each curve is parallel to the tangent planes.

If a series of confocal paraboloids be touched by parallel

planes, it will be found that the points of contact all lie in a bifocal right line, and that the normals at these points lie in a plane parallel to the axis of the surfaces; so that the part of any normal which is intercepted by the two principal planes is constant. This theorem may be proved from the two following properties of the paraboloid:—1. A normal being applied to the surface at the point S, the segments of the normal, measured from S to the points where it intersects the planes of the two principal sections, are to each other inversely as the parameters of these sections. 2. Supposing the axis of x to be that of the surface, the difference between the coordinates x of the point S and of the point where the normal meets the plane of one of the principal sections, is equal to the semiparameter of the other principal section.

§ 11. Let a tangent plane, applied at any point S of a surface of the second order, intersect the plane of one of its focals in the right line Θ, and let P be the foot of the perpendicular dropped from S upon the latter plane. The pole of the right line Θ, with respect to the principal section lying in this plane, is the point P. Let N be its pole with respect to the focal. Then if T be any point of the right line Θ, the polar of this point with respect to the section will pass through P, and its polar with respect to the focal will pass through N; and if the former polar intersect the dirigent curve in Δ, Δ', and the latter intersect the focal in F, F′, the points F, F′ will correspond respectively to the points Δ, Δ', and the distances $\Delta\Delta'$ and FF′ will be similarly divided by the points P and N (See Part I. § 8). But since the point S is in the plane of the two directrices which pass through Δ and Δ', the lengths ΔP and Δ'P, which are the perpendicular distances of S from the directrices, are proportional to the lengths FS and F′S. Therefore FN is to F′N as FS is to F′S, and the right line NS bisects one of the angles made by the right lines FS and F′S. And as this holds wherever the point T is taken on the right line Θ, that is,

in whatever direction the right line FF′ passes through the point N, it follows that the right line NS is an axis of the cone which has the point S for its vertex and the focal for its base. Further, if FF′ intersect Θ in the point Q, we have FN to F′N as FQ is to F′Q, because N is the pole of Θ with respect to the focal; therefore FQ is to F′Q as FS is to F′S, and hence the right line QS also bisects one of the angles made by FS and F′S. The right lines NS and QS are therefore at right angles to each other, and as the latter always lies in the tangent plane, the former must be perpendicular to that plane.

Consequently the normal at any point of a surface of the second order is an axis of the cone which has that point for its vertex and either of the focals for its base.

It is known that when two confocal surfaces intersect each other, they intersect everywhere at right angles; and that through any given point three surfaces may in general be described, which shall have the same focal curves. If three confocal surfaces pass through the point S, the normal to each of them at S is an axis of each of the cones which stand on the focals and have S for their common vertex. The normals to the three surfaces are therefore the three axes of each cone.

If the points at which a series of confocal surfaces are touched by parallel planes be the vertices of cones having one of the focals for their common base, each of these cones will have one of its axes perpendicular to the tangent planes. Therefore when an axis of a cone which stands on a given base is always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line, according as the base is a central conic or a parabola.

§ 12. A system of three confocal surfaces intersecting each other consists of an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, if the focals be central conics; but it consists of two elliptic paraboloids

and a hyperbolic paraboloid, if the focals be parabolas. In the central system, the ellipsoid has the greatest primary axis, and the hyperboloid of two sheets the least; and the focal which is modular in one of these surfaces is umbilicar in the other. The asymptotic cones of the hyperboloids are confocal, the focal lines of each cone being the asymptotes of the focal hyperbola. In the system of paraboloids, the two elliptic paraboloids are distinguished by the circumstance that the modular focal of the one is the umbilicar focal of the other.

The curve in which two confocal surfaces intersect each other is a line of curvature of each, as is well known;* and a series of lines of curvature on a given surface are found by making a series of confocal surfaces intersect it.

Now if a series of the lines of curvature of a given surface be projected on one of its directive planes by right lines parallel to either of its non-directive axes, the projections will be a series of confocal conics; and when the surface is umbilicar, the foci of all these conics will be the corresponding projections of the umbilics.† When the surface is not umbilicar, its directive axis will be parallel to the primary axis of the projections.

The same line of curvature has two projections, according as it is projected by right lines parallel to the one or to the other non-directive axis. In the ellipsoid these projections are always curves of different kinds, the one being an ellipse when the other is a hyperbola; but in a hyperboloid the projections are either both ellipses or both hyperbolas. In the hyperbolic paraboloid the projections are parabolas. In the elliptic paraboloid, one of the projections is always a parabola, and the other is either an ellipse or a hyperbola.

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* See Dupin's Développements de Géométrie.

† Exam. Papers, An. 1838, p. xlvi., quest. 4; p. xcix., quest. 70.

The corresponding projections of two lines of curvature which pass through a given point of the surface, are confocal conics intersecting each other in the projection of that point, and of course intersecting at right angles.

§ 13. A *bifocal chord* is a bifocal right line terminated both ways by the surface.* In a central surface, the length of a bifocal chord is proportional to the square of the diameter which is parallel to it; the square of the diameter being equal to the rectangle under the chord and the primary axis.

More generally, if a chord of a given central surface touch two other given surfaces confocal with it, the length of the chord will be proportional to the square of the parallel diameter of the first surface, the square of the diameter being equal to the rectangle under the chord and a certain right line $2l$, determined by the formula

$$l^2 = \frac{\text{PQR}}{(\text{P} - \text{P}')(\text{P} - \text{P}'')}, \tag{7}$$

wherein it is supposed that the equation (2) represents the first surface, and that P', P'' are the quantities corresponding to P in the equations of the other two surfaces.

In any surface of the second order, the lengths of two bifocal chords are proportional to the rectangles under the segments of any two intersecting chords to which they are parallel.

In the paraboloid expressed by the equation

$$\frac{y^2}{p} + \frac{z^2}{q} = x,$$

if χ be the length of a bifocal chord making the angles β and γ with the axes of y and z respectively, we have

$$\frac{1}{\chi} = \frac{\cos^2\beta}{p} + \frac{\cos^2\gamma}{q}. \tag{8}$$

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* The theorems in § 13 are now stated for the first time.

§ 14. At the point S on a given central surface expressed by the equation (2), let a tangent plane be applied, and let k, k' be the squares of the semiaxes of a central section made in the surface by a plane parallel to the tangent plane; each of the quantities k, k' being positive or negative according as the corresponding semiaxis of the section is real or imaginary, that is, according as it meets the given surface or not. Then the equations* of two other surfaces confocal with the given one, and passing through the point S, are

$$\frac{x^2}{\mathrm{P}-k}+\frac{y^2}{\mathrm{Q}-k}+\frac{z^2}{\mathrm{R}-k}=1, \quad \frac{x^2}{\mathrm{P}-k'}+\frac{y^2}{\mathrm{Q}-k'}+\frac{z^2}{\mathrm{R}-k'}=1. \quad (9)$$

The given surface is intersected by these two surfaces respectively in the two lines of curvature which pass through the point S; the tangent drawn to the first line of curvature at S is parallel to the second semiaxis of the section, and the tangent drawn to the second line of curvature at S is parallel to the first semiaxis of the section.

When two confocal surfaces intersect, the normal applied to one of them at any point S of the line of curvature formed by their intersection lies in the tangent plane of the other, and is parallel to an axis of any section made in the latter by a plane parallel to the tangent plane. Supposing the surfaces to be central, if two normals be applied at the point S, and a diameter of each surface be drawn parallel to the normal of the other, the two diameters so drawn will be equal and of a constant length, wherever the point S is taken on the line of curvature; the square of that length being equal to the difference of the squares of the primary axes of the surfaces, and the diameter of the surface which has the greater primary axis being real, while that of the other surface is imaginary. As the point S moves along the line of curvature, each constant diameter

* Exam. Papers, An. 1837, p. c., quests. 4, 5, 6; An. 1838, p. c., quests. 71, 72.

describes a cone condirective with the surface to which it belongs; the two cones so described are reciprocal, and the focal lines of the cone which belongs to one surface are perpendicular to the directive planes of the other surface.

When two confocal paraboloids intersect, if normals be applied to them at any point S of their intersection, and a bifocal chord of each surface be drawn parallel to the normal of the other, the two chords so drawn will be equal and of a constant length, wherever the point S is taken in the line of intersection of the surfaces; that constant length being equal to the difference between the parameters of either pair of coincident principal sections.

§ 15. The point S being the common intersection of a given system of confocal surfaces, of which the equations are

$$\frac{x^2}{\mathrm{P}} + \frac{y^2}{\mathrm{Q}} + \frac{z^2}{\mathrm{R}} = 1, \qquad \frac{x^2}{\mathrm{P}'} + \frac{y^2}{\mathrm{Q}'} + \frac{z^2}{\mathrm{R}'} = 1,$$
$$\frac{x^2}{\mathrm{P}''} + \frac{y^2}{\mathrm{Q}''} + \frac{z^2}{\mathrm{R}''} = 1, \tag{10}$$

suppose that another surface A confocal with these, and expressed by the equation

$$\frac{x^2}{\mathrm{P}_0} + \frac{y^2}{\mathrm{Q}_0} + \frac{z^2}{\mathrm{R}_0} = 1, \tag{11}$$

is circumscribed by a cone having its vertex at S. If the normals applied at S to the given surfaces, taken in the order of the equations (10), be the axes of new rectangular coordinates ξ, η, ζ, the equation of the cone, referred to these coordinates, will be*

* The equation (12) was obtained in the year 1832, and was given at my lectures in Hilary Term, 1836. The most remarkable properties of cones circumscribing confocal surfaces, are immediate consequences of this equation. That such cones, when they have a common vertex, are confocal, their focal lines being the generatrices of the hyperboloid of one sheet passing through the ver-

$$\frac{\xi^2}{P - P_0} + \frac{\eta^2}{P' - P_0} + \frac{\zeta^2}{P'' - P_0} = 0. \qquad (12)$$

The surfaces of the given system, in the order of their equations, may be supposed to be an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets; the axes of x, y, z being respectively the primary, the mean, and the secondary axes of each surface. Then P is greater than P', and P' greater than P''.

The normals to the given surfaces are the axes of the cone expressed by the equation (12); and if the surface A be changed, but still remain confocal with the given system, it is obvious from that equation that the focal lines of the circumscribing cone will remain unchanged, since the differences of the quantities by which the squares of ξ, η, ζ are divided are independent of the surface A. As P' is intermediate in value between P and P'', the normal to the hyperboloid of one sheet is always the mean axis of the cone; the focal lines lie in the plane $\xi\zeta$, and their equation is

$$\frac{\xi^2}{P' - P} + \frac{\zeta^2}{P' - P''} = 0, \qquad (13)$$

which shows that they are parallel to the asymptotes of a central section made in the hyperboloid of one sheet by a plane parallel to the plane $\xi\zeta$, since the quantities $P' - P$ and $P' - P''$ are (including the proper signs) the squares of the semiaxes of the section which are parallel to ξ and ζ re-

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tex, was first stated by Professor C. G. J. Jacobi, of Königsberg, in 1834. See Crelle's Journal, vol. xii., p. 137. See also the excellent work of M. Chasles, published in 1837, and entitled "Aperçu historique sur l'Origine et le Développement des Méthodes en Géométrie;" p. 387. The analogy which exists between the focals of surfaces and the foci of curves of the second order was supposed by M. Chasles to have been pointed out in that work for the first time (*Comptes rendus*, tom. xvi., pp. 833, 1106); but that analogy had been previously taught and developed in the lectures just alluded to.

spectively. The focal lines are therefore the generatrices of that hyperboloid at the point S.

When $R_0 = 0$, the equation (12) becomes

$$\frac{\xi^2}{R} + \frac{\eta^2}{R'} + \frac{\zeta^2}{R''} = 0, \tag{14}$$

which is that of the cone standing on the focal ellipse and having its vertex at S. When $Q_0 = 0$, the same equation becomes

$$\frac{\xi^2}{Q} + \frac{\eta^2}{Q'} + \frac{\zeta^2}{Q''} = 0, \tag{15}$$

which is that of the cone standing on the focal hyperbola, and having its vertex at S. The normal to the hyperboloid of one sheet at the point S is the mean axis of both cones; the normal to the ellipsoid is the internal axis of the first cone and the directive axis of the second, while the normal to the hyperboloid of two sheets is the directive axis of the first and the internal axis of the second.

The three surfaces expressed by the equations

$$\frac{\xi^2}{P} + \frac{\eta^2}{P'} + \frac{\zeta^2}{P''} = 1, \qquad \frac{\xi^2}{Q} + \frac{\eta^2}{Q'} + \frac{\zeta^2}{Q''} = 1,$$
$$\frac{\xi^2}{R} + \frac{\eta^2}{R'} + \frac{\zeta^2}{R''} = 1, \tag{16}$$

are a confocal system, having their centre at S, and being respectively an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. They intersect each other in the centre of the system expressed by the equations (10), and their normals at that point are the axes of x, y, z respectively. The relations between the two systems of surfaces are therefore perfectly reciprocal. From the equations (14) and (15) it is manifest that the asymptotic cones of the hyperboloids of one system pass through the focals of the other.

§ 16. The point S being the intersection of a given system of confocal paraboloids whose equations are

$$\frac{y^2}{p} + \frac{z^2}{q} = x + h, \qquad \frac{y^2}{p'} + \frac{z^2}{q'} = x + h',$$
$$\frac{y^2}{p''} + \frac{z^2}{q''} = x + h'', \tag{17}$$

where $p - p' = q - q' = 4(h - h')$, and $p - p'' = q - q''$ $= 4(h - h'')$; suppose that another paraboloid A confocal with these, and expressed by the equation

$$\frac{y^2}{p_0} + \frac{z^2}{q_0} = x + h_0, \tag{18}$$

is circumscribed by a cone having its vertex at S. Then if the normals applied at S to the given system of surfaces, taken in the order of their equations, be the axes of the coordinates ξ, η, ζ respectively, the equation of the circumscribing cone will be

$$\frac{\xi^2}{p - p_0} + \frac{\eta^2}{p' - p_0} + \frac{\zeta^2}{p'' - p_0} = 0; \tag{19}$$

showing that those normals are the axes of the cone, and that the focal lines of the cone are independent of the surface A, provided it be confocal with the given surfaces. If the hyperbolic paraboloid be the second surface of the given system, the parameter p' will be intermediate in value between p and p'', and the equation of the focal lines of the cone will be

$$\frac{\xi^2}{p' - p} + \frac{\zeta^2}{p' - p''} = 0, \tag{20}$$

which is the equation of a pair of right lines parallel to the asymptotes of a section made in the hyperbolic paraboloid by a plane parallel to the plane $\xi\zeta$, since the quantities $p' - p$ and $p' - p''$ are proportional to the squares of the semiaxes of the section which are parallel to ξ and ζ respectively. The focal lines are therefore the generatrices of the hyperbolic paraboloid at the point S.

Putting p_0 and q_0 alternately equal to zero in the equation (19), we get

$$\frac{\xi^2}{p}+\frac{\eta^2}{p'}+\frac{\zeta^2}{p''}=0, \qquad \frac{\xi^2}{q}+\frac{\eta^2}{q'}+\frac{\zeta^2}{q''}=0, \qquad (21)$$

the equations of two cones which have a common vertex at S, the first of them standing on the focal which lies in the plane xz, the second on the focal which lies in the plane xy. The mean axis of each of these cones is the normal at S to the hyperbolic paraboloid; the internal axis of either cone is the normal to the elliptic paraboloid which has the base of that cone for its modular focal.

As the cones which have a common vertex, and stand on the focals of any surface of the second order, are confocal, they intersect at right angles. Therefore when two planes passing through a bifocal right line touch the focals, these planes are at right angles to each other. And as cones which have a common vertex, and circumscribe confocal surfaces, are confocal, two such cones, when they intersect each other, intersect at right angles. Therefore when a right line touches two confocal surfaces, the tangent planes passing through this right line are at right angles to each other.

§ 17. When two surfaces are reciprocal polars* with respect to any sphere, and one of them is of the second order, the other is also of the second order. Let the surface B be reciprocal to the surface A before mentioned, with respect to a sphere of which the centre is S; and suppose R′ and R to be any corresponding points on these surfaces. Then the plane which touches the surface A at the point R, intersects the right line SR′ perpendicularly in a point K, such that the rectangle under SR′ and SK is constant, being equal to the

* Transactions of the Royal Irish Academy, vol. xvii., p. 241; Exam. Papers, An. 1841, p. cxxvi., quest. 4.

square of the radius of the sphere. Now if the point K approach indefinitely to S, the distance SR′ will increase without limit, the surface B being of course a hyperboloid; and if through S any plane be drawn touching the surface A, a right line perpendicular to this plane will evidently be parallel to a side of the asymptotic cone of the hyperboloid. The asymptotic cone of B is therefore reciprocal to the cone which, having its vertex at S, circumscribes the surface A. Hence, as the directive planes of a hyperboloid are the same as those of its asymptotic cone, it follows that the directive planes of the surface B are perpendicular to the generatrices of the hyperboloid of one sheet, or the hyperbolic paraboloid, which passes through S, and is confocal with the surface A. And this relation between two reciprocal surfaces ought to be general, whatever be the position of the point S with respect to them;* for though it has been deduced by the aid of the circumscribing cone aforesaid, it does not, in its enunciation, imply the existence of such a cone. This conclusion may be verified by investigating the equation of the surface B in terms of the coordinates ξ, η, ζ. Suppose ρ to be the radius of the sphere with respect to which the surfaces A and B are reciprocal. Then if A be a central surface expressed by the equation (11), and having ξ_0, η_0, ζ_0 for the coordinates of its centre, the surface B will be represented by the equation

$$\begin{aligned}(\mathrm{P}-\mathrm{P}_0)\,\xi^2 + (\mathrm{P}'-\mathrm{P}_0)\,\eta^2 + (\mathrm{P}''-\mathrm{P}_0)\,\zeta^2 \\ = 2\rho^2(\xi_0\xi + \eta_0\eta + \zeta_0\zeta) - \rho^4;\end{aligned} \tag{22}$$

but if A be a paraboloid expressed by the equation (18), the equation of B will be

$$\begin{aligned}(p-p_0)\,\xi^2 + (p'-p_0)\,\eta^2 + (p''-p_0)\,\zeta^2 \\ = 4\rho^2(\xi\cos\alpha + \eta\cos\beta + \zeta\cos\gamma),\end{aligned} \tag{23}$$

where α, β, γ are the angles which the axis of x makes with

* This relation was first noticed by Mr. Salmon.

the axes of ξ, η, ζ respectively. In the first case, the equation (22) shows that the directive planes of B are perpendicular to the right lines expressed by the equation (13); in the second case, the equation (23) shows that the directive planes of B are perpendicular to the right lines expressed by the equation (20).

When the surface A is a paraboloid, and the distance of the point R from its vertex is indefinitely increased, the plane touching the surface at R approaches indefinitely to parallelism with its axis, and the right line SK, perpendicular to that plane, increases without limit. Therefore the surface B passes through the point S, and is touched in that point by a plane perpendicular to the axis of A.

When the point S lies upon the surface A, the coefficient of the square of one of the variables, in the equation (22) or (23), is reduced to zero, and the surface B is a paraboloid having its axis parallel to the normal applied at S to the surface A. This also appears from considering that when S is a point of the surface A, the normal at that point is the only right line passing through S, which meets the surface B at an infinite distance.

If a series of surfaces be confocal, their reciprocal surfaces, taken with respect to any given sphere, will be condirective.

When the equations of any two condirective surfaces are expressed by coordinates perpendicular to their principal planes, the constants in the equations may be always so taken that the differences of the coefficients of the squares of the variables in one equation shall be equal to the corresponding differences in the other. Then by subtracting the one equation from the other, we get the equation of a sphere. Therefore when two condirective surfaces intersect each other, their intersection is, in general, a spherical curve. But when the surfaces are two paraboloids of the same species, their intersection is a plane curve.

§ 18. Through any point S of a given surface four bifocal right lines may in general be drawn. Supposing the surface to be central, let a plane drawn through the centre, parallel to the plane which touches the surface at S, intersect any one of these right lines. Then the distance of the point of intersection from the point S will always be equal to the primary semiaxis of the surface.*

If through any point S of a given central surface a right line be drawn touching two other given surfaces confocal with it, and if this right line be intersected by a plane drawn through the centre parallel to the plane which touches the first surface at S, the distance of the point of intersection from the point S will be constant, wherever the point S is taken on the first surface. If this constant distance be called l, and the other denominations be the same as in the formula (7), the value of l will be given by that formula.†

Professor Mac Cullagh communicated the following note relative to the comparison of arcs of curves, particularly of plane and spherical conics.

The first Lemma given in my paper on the rectification of the conic sections (Transactions of the Royal Irish Academy, vol. xvi., p. 79) is obviously true for curves described on any given surface, provided the tangents drawn to these curves be *shortest lines* on the surface. The demonstration remains exactly the same; and the Lemma, in this general form, may be stated as follows.

Understanding a tangent to be a shortest line, and supposing two given curves E and F to be described on a given

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* Exam. Papers, An. 1838, p. xlvii., quest. 9.

† In the notes to the last mentioned work of M. Chasles, on the History of Methods in Geometry, will be found many theorems relative to surfaces of the second order. Among them are some of the theorems which are given in the present paper; but it is needless to specify these, as M. Chasles's work is so well known.